Contemporary Mathematics
Volume **271**, 2001

MAPPING CLASS GROUPS AND FUNCTION SPACES

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1. Introduction

The purpose of this paper is to explore a relationship between mapping class groups and certain function spaces. One application is a determination of the odd primary cohomology of the "hyperelliptic mapping class group" to be defined below together with the cohomology of the mapping class group for a punctured 2-sphere.

Let $M^n_{g,k}$ denote a Riemann surface of genus g with k boundary components and n punctures. The mapping class group $\Gamma^n_{g,k}$ is $\pi_0 diff^+(M^n_{g,k})$ where $diff^+(\cdot)$ is the group of orientation preserving diffeomorphisms, possibly permuting punctures, fixing the boundary circles pointwise.

The surface $M_g = M^0_{g,0}$ can be regarded as a 2-sheeted branched cover $\pi : M_g \to S^2$ with $2g+2$ branch points as follows. There is an involution $j : M_g \to M_g$ with $2g+2$ fixed points and S^2 is the orbit space M_g/π where π is the group $\mathbb{Z}/2\mathbb{Z}$ acting through j. Define the hyperelliptic mapping class group $\Delta_g \leq \Gamma^0_{g,0}$ to be the subgroup of Γ_g which is the centralizer of the isotopy class group of j. For example, $\Delta_2 = \Gamma_2$ but Δ_g is a proper subgroup of Γ_g if $g \geq 3$. In addition there is a non-split central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Delta_g \to \Gamma^{2g+2} \to 1$$

where $\Gamma^n = \Gamma^n_{0,0}$. References for these facts are [3], [4], [5].

Thus the map $\Delta_g \to \Gamma^{2g+2}$ gives a cohomology isomorphism with coefficients in a ring R containing $1/2$. The methods here apply to the groups Γ^n rather than the groups Δ_g directly. If g is even, the 2-primary cohomology of the groups Δ_g is analyzed in [14] using more geometric techniques. The main results here are a determination of the mod-p cohomology of all of the Γ^n for p an odd prime with either trivial coefficients, or coefficients in the sign representation.

To describe the relation between Γ^n and function spaces, consider the space $\Lambda^k X$ of all continuous maps from the k-sphere, S^k, to X. The group $O(k+1)$ acts naturally on S^k by reflections and thus on the space $\Lambda^k X$. If G is a subgroup of $O(k+1)$, consider the homotopy orbit space (Borel construction), $ESO(k+1) \times_G \Lambda^k X$. It follows from what is done below that if X is a $(k+1)$-fold suspension, then fibrewise analogues of configuration spaces with labels in X provide combinatorial models for $ESO(k+1) \times_G \Lambda^k X$. When $k = 2$, and $G = SO(3)$, the "building blocks" for this model are Eilenberg-Mac Lane spaces $K(\Gamma^n, 1)$ much in the same way that the configuration spaces $F(\mathbb{R}^2, n)/\Sigma_n$ are building blocks for double loop spaces of double suspensions.

Date: June 2000.

*Partially supported by an NSF grant and by the S.F.B. 170 in Göttingen during the preparation of this article.

The cohomology for some of these function spaces can be analyzed by inspection of the Serre spectral sequence for the fibrations

1. $\Lambda^2 X \to ESO(3) \times_{SO(3)} \Lambda^2 X \to BSO(3)$, and
2. $\Omega^2 X \to \Lambda^2 X \to X$.

The cohomology of the groups Γ^n follows directly from the cohomology of these function spaces. The approach above may be regarded as using "continuous information" to obtain information about the cohomology of discrete groups. This point of view shares some features in common with the cyclic homology for certain algebras. This connection is described in [15]. More detailed descriptions as well as further applications are given in [17].

Some information about Γ^n at the prime p is given in [2] where the mod-2, 3, and 5 cohomology of Γ^6 was first given. Related interesting results for $Sp(4,\mathbb{Z})$ are given in [10], [28]. The 2-torsion in the mapping class group for genus 2 as well as that for the hyperelliptic mapping class group Δ_g was given in [14]. For example, there are copies of $\mathbb{Z}/8\mathbb{Z}$ in every positive dimension which is congruent to zero modulo 4; the remaining 2-torsion is all of order 2.

The current article gives a calculation of the mod-p cohomology of Γ^n for all n. For example, Theorem 2.4 lists the mod-2 cohomology of each group Γ^n. Theorem 2.5 gives the mod-p cohomology with trivial coefficients, and with coefficients in the sign representation. This article appeared in mildly different form in the Mathematica Göttingensis preprint series, Heft 5 (1989), [7].

Subsequently, Kawazumi gave related, and interesting information about the mod-p cohomology of these groups using different methods in the special cases for which $p < 2g+2$ [27]. For example, he shows that if $p = g+1$, then the mod-p cohomology ring of Γ^{2g+2}, and Δ_g is a free module over a polynomial ring on one indeterminate of degree 4 with the generator given by the second Morita-Mumford class, and with Euler-Poincaré series $(1+t^3+t^{2g}+t^{2g+1})(1-t^4)^{-1}$. These special cases, and the results given here agree.

The point of this article is a complete determination of the mod-p cohomology of Γ^n for all p and for all n in a global and simple form. This same structure is used elsewhere to give analogous results for genus one surfaces with punctures (in preparation).

The models given here were, of course, motivated by constructions originally given by Dyer and Lashof [19], and by many others [1, 6, 24, 25, 26, 29, 30, 31, 32]. Although this article was written some time ago [7], the authors had decided to wait to publish the results. The article was submitted now because the mathematics fits some of the topics in which Peter is interested.

We wish him all the best on this happy occasion of his 60-th birthday.

2. Results

Let $\pi : E \to B$ denote the projection map for a fibre bundle with fibre Y. Configuration spaces were defined by Fadell and Neuwirth [20] and bundle analogues were studied in [18] as follows:

(1): $F(Y,k)$ is the subspace of Y^k given by $\{(y_1,\dots,y_k) \mid y_i \neq y_j \text{ if } i \neq j\}$ and
(2): $E(\pi,k) = \{(e_i,\dots,e_k) \in E^k \mid e_i \neq e_j \text{ and } \pi(e_i) = \pi(e_j) \text{ if } i \neq j\}$.

Thus there are bundles

$$F(Y,k) \to E(\pi,k) \xrightarrow{p} B$$

and

$$F(Y,k)/\Sigma_k \to E(\pi,k)/\Sigma_k \xrightarrow{p} B$$

where π is first coordinate projection and the symmetric group Σ_k acts by permuting coordinates.

Consider the inclusion of $SO(2)$ in $SO(3)$ and the induced map $\eta : BSO(2) \longrightarrow BSO(3)$ to get a bundle $S^2 \xrightarrow{*} BSO(2) \xrightarrow{\eta} BSO(3)$. Several proofs of the following proposition appear in [14, 2] although this result was known for some time. A third proof is included for the convenience of the reader.

Proposition 2.1. *If $k \geq 3$, then $E(\eta,k)$ is an Eilenberg-Mac Lane space $K(\pi,1)$ and $E(\eta,k)/\Sigma_k$ is the Eilenberg-Mac Lane space $K(\Gamma^k,1)$. Thus there are isomorphisms*

- **(i):** *$H^*(\Gamma^k; S) \cong H^*(E(\eta,k)/\Sigma_k; S)$ for any trivial Γ^k-module S and*
- **(ii):** *$H^*(\Delta_g; R) \cong H^*(E(\eta,2g+2)/\Sigma_{2g+2}; R)$ where R is any ring which is a trivial Δ_g-module containing 1/2.*

The groups Γ^k are related to function spaces via a combinatorial construction. Let X denote a connected CW-complex with base-point $*$. Write

$$E(\pi;X) = \coprod_{k\geq 0} E(\pi,k) \times_{\Sigma_k} X^k/(\approx)$$

where $\approx$ is the equivalence relation determined by

$$(e_i,\ldots,e_k)(x_1,\ldots,x_k) \approx (e_1,\ldots,\hat{e}_i,\ldots,e_k)(x_1,\ldots,\hat{x}_i,\ldots,x_k)$$

provided $x_i = *$. Thus $E(\pi;X)$ is the equivalence classes of pairs $[S,f]$ where S is a finite subset of E with $\pi(S) = \{\text{one point}\}$ and $f : S \to X$ where $[S-\{q\}, f|\{S-\{q\}\}]$ is equivalent to $[S,f]$ provided $f(q) = *$. In case B is a point, the notation $C(Y;X)$ for $E(\pi;X)$ is that of [18, 12, 16].

As a special case, let $\pi : BSO(n) \longrightarrow BSO(n+1)$ be given by the standard inclusion of $SO(n)$ in $SO(n+1)$. Thus the fibre of π is S^n. Recall that $SO(n+1)$ acts on S^n as usual and so $SO(n+1)$ acts naturally on $\Lambda^n X$, the space of all continuous maps from S^n to X.

Proposition 2.2. *If X is a connected CW-complex, there is a homotopy equivalence*

$$E(\pi;\Sigma X) \longrightarrow ESO(n+1) \times_{SO(n+1)} \Lambda^n \Sigma^{n+1} X.$$

The spaces $E(\pi;X)$ are filtered by the cardinality of S in an element $[S,f]$ as follows. Write $E_k(\pi;X)$ for the k-th filtration and $D_k(\pi;X)$ for the filtration quotient $E_k(\pi;X)/E_{k-1}(\pi;X)$. The next proposition is not stated explicitly in [12, 18], and elsewhere but nevertheless, it follows immediately via the proof in the appendix of [12].

Proposition 2.3. *If X is a connected CW-complex, then $E(\pi;X)$ is stably equivalent to $\bigvee_{k\geq 1} D_k(\pi;X)$. Thus there is an isomorphism*

$$\bar{H}_* E(\pi;X) \cong \bigoplus_{k\geq 1} \bar{H}_* D_k(\pi;X)\ .$$

Write B_n for Artin's braid group on n strings. That the cohomology of Γ^n is a module over the cohomology of $BSO(3)$ will be used next.

Theorem 2.4. 1. *If $n \geq 2$, there is an isomorphism of $H^*(BSO(3);\mathbb{F}_2)$-modules*

$$H^*(\Gamma^{2n};\mathbb{F}_2) \cong H^*(BSO(3);\mathbb{F}_2) \otimes H^*(F(S^2,2n)/\Sigma_{2n};\mathbb{F}_2)$$

and $H^*(F(S^2,2n)/\Sigma_{2n};\mathbb{F}_2)$ is isomorphic to $H^*(B_{2n};\mathbb{F}_2) \oplus H^{*-2}(B_{2n-2};\mathbb{F}_2)$ as a vector space.

2. *If $n \geq 2$, there is an isomorphism of $H^*(BSO(2);\mathbb{F}_2)$-modules*

$$H^*(\Gamma^{2n-1};\mathbb{F}_2) \cong H^*(BSO(2);\mathbb{F}_2) \otimes H^*(B_{2n-2};\mathbb{F}_2)\,.$$

The isomorphisms in Theorem 2.4 frequently do not preserve the algebra structure. For example, if x is the non-zero element of $H^1(\Gamma^{2n};\mathbb{F}_2)$, then the cup-square x^2 is non-trivial. In the mod-2 cohomology of the braid groups, the cup-square of the one dimensional class is zero as seen in [11, 22, 33] where the cohomology of the braid groups is given.

The methods also give information about the cohomology of Γ^n with other coefficients. For example, let V be a graded vector space over a field $\mathbb{F}$ and write $V^{\otimes n}$ for the Γ^n-module $\underbrace{V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V}_{n}$ where Γ^n acts via Σ_n permuting coordinates with the usual sign conventions. Assume that V is concentrated in degrees at least one. Let S_V denote a bouquet of spheres with $\bar{H}_*(S_V;\mathbb{F})$ isomorphic to V. Then there is an isomorphism in section 3 given by

$$H_*(\Gamma^n : V^{\otimes n}) \cong H_*(D_n(\eta;S_V);\,\mathbb{F})$$

for $n \geq 3$ and η is given in Proposition 2.1. In particular, if V is $\mathbb{F}$ concentrated in an odd degree, then $V^{\otimes n}$ is the sign representation which will be written $\mathbb{F}(-1)$.

To state the next theorem, write $D_k(\mathbb{R}^2;S^n)$ and $D_k(S^2;S^n)$ for the filtration quotients of $\Omega^2 S^{n+2}$ and $\Lambda^2 S^{n+2}$ respectively. The homology of $D_k(\mathbb{R}^2;S^n)$ is given in [11] while that of $\Lambda^2 S^N$ is given in section 8 here. Homology groups in the next theorem are taken with $\mathbb{F}_p$-coefficients.

Theorem 2.5. 1. *If p is an odd prime, there are isomorphisms of vector spaces*

$$H_*E(\eta;S^{2n-1}) \cong H_*BS^3 \otimes H_*\Lambda^2 S^{2n+1}$$

and

$$H_q(\Gamma^k;\mathbb{F}_p(-1)) \cong \Sigma_i H_i BS^3 \otimes \bar{H}_{q-i-k(2n-1)} D_k(S^2;S^{2n-1})\,.$$

2. *If $p \geq 3$, $H_q(\Gamma^k;\mathbb{F}_p)$ will be given in section 10 here.*

The answers in Theorem 2.5(2) above require more explicit information, and are given in Theorems 10.1, and 10.3.

Some previous results that were given in [2] are as follows. It was shown that the integral cohomology of Γ^n is all p-torsion for $p \leq n$. One curious example is $K(\Gamma^4,1)$ which stably splits as a wedge $A \vee B$ where (1) A is $K(\Sigma_4,1) \vee \Sigma(M(2))$ localized at 2 and $\Sigma M(2)$ is the homotopy direct limit of the second Steinberg idemipotent $e_2 : \Sigma RP^\infty \wedge RP^\infty \longrightarrow \Sigma RP^\infty \wedge RP^\infty$ with

$$e_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

and (2) $B = K(\mathbb{Z}/3\mathbb{Z},1)$. The mod-2,3, and 5 cohomology of Γ^6 was also given. These special cases agree with Theorems 2.4 and 2.5. The map $B\Gamma^n \longrightarrow BSO(3)$ of Lemma 2.1 does not arise as the classifying map of a group homomorphism if $n \geq 6$.

The function space methods here apply to give the cohomology of a few other discrete groups related to Γ_g. For example, the rational cohomology groups of Γ_1^k with either trivial rational coefficients or coefficients in the sign representation have a succinct description in terms of classical modular forms.

These results lead to the following (somewhat imprecise) question. Can one study the cohomology of certain discrete groups such as $\Gamma_{g,1}$ or the automorphism group of the free group by variations on these function space methods ?

3. $\Gamma^n, SO(3)$ AND PROPOSITION 2.1

Throughout this section, the bundle projection $BSO(2) \to BSO(3)$ with fibre S^2 given by the standard inclusion of $SO(2)$ in $SO(3)$ will be denoted by η.

Proposition 3.1. *If $k \geq 3$, then $E(\eta,k)/\Sigma_k$ is an Eilenberg-Mac Lane space $K(\Gamma^k, 1)$.*

To prove 3.1, first check that $\pi_i E(\eta, k) = 0$ if $i > 1$ and $k \geq 3$. Here consider the diagram of fibrations where p denotes first coordinate projection, and i denotes the natural inclusion [18].

$$\begin{array}{ccccc}
F(\mathbb{R}^2, k-1) & \longrightarrow & F(S^2, k) & \xrightarrow{p} & S^2 \\
\downarrow & & \downarrow & & \downarrow i \\
F(\mathbb{R}^2, k-1) & \longrightarrow & E(\eta, k) & \xrightarrow{p} & BSO(2) \\
\downarrow & & \downarrow & & \downarrow \\
\{*\} & \longrightarrow & BSO(3) & \longrightarrow & BSO(3)
\end{array}$$

Since $F(\mathbb{R}^2, k-1)$ is a $K(\pi, 1)$, it follows that $E(\eta, k)$ is also provided the boundary homomorphism

$$\partial : \pi_2 BSO(2) = \pi_2 \mathbb{C}P^\infty \to \pi_1 F(\mathbb{R}^2, k-1)$$

is a monomorphism. Since i induces a monomorphism on π_2 and $F(\mathbb{R}^2, k-1)$ is a finite dimensional $K(\pi, 1)$, it suffices to check that the boundary homomorphism

$$\partial : \pi_2 S^2 \to \pi_1 F(\mathbb{R}^2, k-1)$$

is non-zero. If $k = 3$, this follows directly from the fact that $F(S^2, 3)$ is homotopy equivalent to $\mathbb{R}P^3$. If $k \geq 3$, a similar result follows by comparison of the fibrations where q denotes s projection on the first 3 coordinates.

$$\begin{array}{ccccc}
F(\mathbb{R}^2, k-1) & \longrightarrow & F(S^2, k) & \xrightarrow{p} & S^2 \\
\downarrow & & q \downarrow & & \downarrow i \\
F(\mathbb{R}^2, 2) & \longrightarrow & F(S^2, 3) & \xrightarrow{p} & S^2
\end{array}$$

These observations are recorded next.

Lemma 3.2. *If $k \geq 3$, $E(\eta, k)$ is a $K(\pi, 1)$.*

To finish the proof of 3.1, it suffices to identify the fundamental group of $E(\eta, k)/\Sigma_k$. Consider the following map of fibrations to get a long exact sequence in homotopy given by

$$0 \to \pi_2 BSO(3) \to \pi_i F(S^2, k)/\Sigma_k \to \pi_1 E(\eta, k)/\Sigma_k \to \pi_1 BSO(3) \to 0$$

from the following morphisms of fibrations:

$$\begin{array}{ccccc}
F(S^2, k)) & \longrightarrow & E(\eta, k) & \xrightarrow{p} & BSO(3) \\
\downarrow & & \downarrow & & \downarrow i \\
F(S^2, k)/\Sigma_k & \longrightarrow & E(\eta, k)/\Sigma_k & \xrightarrow{p} & BSO(3) \\
\downarrow & & \downarrow & & \downarrow \\
B\Sigma_k & \longrightarrow & B\Sigma_k & \longrightarrow & \{*\}
\end{array}$$

Since the center of $\pi_1 F(S^2, k)/\Sigma_k$ is $\mathbb{Z}/2\mathbb{Z}$ and the quotient by the center is isomorphic to Γ^k [21], it follows that $\pi_1 E(\eta, k)/\Sigma_k$ is isomorphic to Γ^k.

The above procedure applies to certain other fibre bundles with fibre given by a surface [15]. Applications have been given to the cohomology of some of these groups, and their relation to modular forms.

4. Mapping spaces and group cohomology

In this section, the cohomology of certain groups with possibly twisted coefficients is considered with the view that the cohomology of certain mapping spaces informs on the cohomology of these groups with the given coefficients. The methods are some of those in [11] and this section is an exposition of those methods.

Let V denote a graded vector space over a field $\mathbb{F}$ and assume that V is concentrated in degrees at least one. Thus there is a bouquet of spheres S_V and an isomorphism of vector spaces

$$V \cong \bar{H}_*(S_V; \mathbb{F}) .$$

The symmetric group Σ_k acts on

$$V^{\otimes k} = V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V$$

by permuting coordinates with the usual sign conventions.

For example, if V is a copy of $\mathbb{F}$ concentrated in even degrees, then $V^{\otimes k}$ is the trivial $\mathbb{F}\Sigma_k$-module. If V is a copy of $\mathbb{F}$ concentrated in odd degrees, then $V^{\otimes k}$ is a copy of the sign representation. As Γ^k maps to Σ_k, $V^{\otimes k}$ is also a Γ^k-module and $H_*(\Gamma^k; V^{\otimes k})$ is $\mathrm{Tor}_*^{\mathbb{F}\Gamma^k}(\mathbb{F}; V^{\otimes k})$.

Proposition 4.1. *If $\eta : BSO(2) \to BSO(3)$ is the bundle projection induced by the natural inclusion of $SO(2)$ in $SO(3)$, then there is a vector space isomorphism for $k \geq 3$ given by*

$$H_*(\Gamma^k; V^{\otimes k}) \cong \bar{H}_*(D_k(\eta; S_V); \mathbb{F}) .$$

Furthermore, there are vector space isomorphisms

$$H_n(\Gamma^k; \mathbb{F}) \cong H_{n+2qk}(D_k(\eta; S^{2q}); \mathbb{F})$$

and

$$H_n(\Gamma^k; \mathbb{F}(\pm 1)) \cong H_{n+(2q+1)k}(D_k(\eta; S^{2q+1}); \mathbb{F})$$

where $\mathbb{F}(\pm 1)$ *denotes the sign representation of* Γ^k.

Since $E(\eta; S_V)$ is stably equivalent to $\bigvee_{k\geq 1} D_k(\eta; S_V)$, the identification of the homology of $E(\eta; S_V)$ as a filtered vector immediately gives $H_*(\Gamma^k; V^{\otimes k})$. Explicit classes are given in later sections of this article.

The proof of 4.1 is standard, and some details are given for the convenience of the reader. Let M be a G-space. Thus there is a principal G-bundle

$$\lambda : EG \times_G M \to BG$$

with fibre M. Then consider

$$E(\lambda, X) = EG \times_G C(M; X)$$

where X is a pointed space and $C(M;X) = E(*;X)$ as given in section 2. Also, consider the spaces

$$E(\lambda, k) = EG \times_G F(M,k)$$

and

$$E(\lambda, k)/\Sigma_k = EG \times_G (F(M,k)/\Sigma_k)$$

where G acts on the configuration space and EG diagonally. By Proposition 2.3, $E(\lambda, X)$ is stably equivalent to $\vee_{k\geq 1} D_k(\lambda; X)$ for connected CW-complexes X.

Let $S_*(X)$ denote the singular chains complex of X over $\mathbb{F}$. Thus there is a homology isomorphism

$$j : H_*(X;\mathbb{F}) \to S_*(X)$$

where $H_*(X;\mathbb{F})$ has the trivial differential and consequently there is a homology isomorphism

$$S_*E(\eta, k) \otimes_{\Sigma_k} (H_*X)^{\otimes k} \to S_*(E(\eta, k) \times_{\Sigma_k} X^k) .$$

If X_k denotes the fat wedge in X^k and X is a path-connected space with non-degenerate base-point, there is a cofibration

$$E(\eta, k) \times_{\Sigma_k} X_k \to E(\eta, k) \times_{\Sigma_k} X^k \to D_k(\eta, X) .$$

Thus there is a homology isomorphism

$$S_*E(\eta, k) \otimes_{\Sigma_k} (\bar{H}_*X)^{\otimes k} \to S_*(E(\eta, k) \times_{\Sigma_k} X^{(k)}/E(\eta, k)\times_{\Sigma_k})$$

where $X^{(k)}$ is the k-fold smash product and 4.1 follows.

5. Stiefel manifolds and the proof of 2.2

Write $V_{n,k}$ for the space of orthonormal k-frames in $\mathbb{R}^n$ and recall the natural $O(n)$-action on $V_{n,k}$ together with the homeomorphisms $SO(n)/SO(n-k) \to O(n)/O(n-k) \to V_{n,k}$. The spaces are used next to obtain new bundles where the main case below is given by η.

For $q \geq 1$, and X a space with base-point, form the configuration space bundle $E(\eta; q)$ and $E(\eta; X)$. There are homeomorphisms

$$E(\eta; q) \to ESO(n) \times_{SO(n)} F(V_{n,k}, q),$$

and

$$E(\eta; X) \to ESO(n) \times_{SO(n)} C(V_{n,k}, X).$$

The homotopy type of some of these spaces are determined next in terms of function spaces. In what follows restrict to the case for which M is closed.

Choose a Riemannian metric on M, and consider the associated exponential map $\exp : D(M) \to M$, defined on the maximal open disc bundle $D(M) \subseteq T(M)$ in the tangent bundle $T(M)$, such that the restriction $\exp_y : D_y(M) \to M$ of exp to a fibre over $y \in M$ is a diffeomorphism onto a neighborhood of y. Then define $\dot{T}(M)$ by identifying two vectors in $T(M)$ if they lie in the same fibre and outside $D(M)$. Thus $\dot{T}(M) \to M$ is a $\mathbb{D}^m/\partial\mathbb{D}^m$-bundle with section. Its' isomorphism type is independent of the metric chosen.

Using the base-point in X there is the naturally defined section for $M \times X \to M$. Form the fibrewise smooth product $\dot{T}(M) \bigwedge_M (M \times X) = \dot{T}(M; X)$, which is a $\Sigma^m X$-bundle over M, where m is the dimension of M. Let $\Gamma(M; X)$ denote the space of sections for this bundle. If $\xi \in C(M; X)$ and $y \in M$ is given, restrict ξ to the neighborhood $\exp(D_y(M))$ by considering the image of ξ under the natural map $C(M; X) \to C(M, M - \exp(D_y(M)); X)$. Notice that there are homeomorphisms

$$C(M, M-\exp(D_y(M)); X) \to C(\exp(D_y(M)), \partial\exp(D_y(M)); X) \to C(\mathbb{D}^m, \partial\mathbb{D}^m; X).$$

The last space is retractible onto its subspace

$$\Sigma^m X = (\mathbb{D}^m/\partial\mathbb{D}^m) \wedge X \subset C(\mathbb{D}^m, \partial\mathbb{D}^m; X)\ .$$

Setting $\gamma(\xi)(y) = \xi_y$, gives a section $\gamma(\xi)$ of the bundle $\dot{T}(M; X)$. The following proposition concerning these constructions and their relationship to spaces of sections is given in [6].

Proposition 5.1. *For a compact, closed, smooth manifold M and a connected CW-complex X the map*

$$\gamma : C(M; X) \to \Gamma(M; X)$$

is a homotopy equivalence.

If M is parallelizable, this gives a homotopy equivalence $C(M; X) \simeq \text{map}\,(M; \Sigma^m X)$, the free maps from M to X. The next result will use instead that $M \times \mathbb{R}^k$ is parallelizable. In this case, there are bundle isomorphisms $T(M) \oplus \epsilon_k \simeq M \times \mathbb{R}^{m+k}$ for the trivial k-plane bundle ϵ_k. Hence, if X' is a k-fold suspension $X' = \Sigma^k X$, then

$$\dot{T}(M; X') = \dot{T}(M) \wedge_M (M \times \Sigma^k X), \textit{and}$$

$$\dot{T}(M) \wedge_M \epsilon_k \wedge_M (M \times X) \cong M \times \Sigma^{m+k} X\ .$$

Thus $\Gamma(M; \Sigma^k X)$ is homotopy equivalent to $map(M; \Sigma^{m+k} X)$.

Corollary 5.2. *For a connected CW-complex X, there is a homotopy equivalence*

$$C(S^m; \Sigma X) \to \ map\ (S^m; \Sigma^{m+1} X) \cong \Lambda^m \Sigma^{m+1} X\ .$$

A diffeomorphism $\phi : M \to M$ acts on $C(M; X)$ and on $\Gamma(M; X)$ in the natural way. If ϕ is an isometry with respect to the metric chosen above, then γ commutes with the induced map ϕ^* on $C(M; X)$ and $\Gamma(M; X)$. Thus the following diagram commutes:

$$\begin{array}{ccc} C(M;X) & \xrightarrow{\gamma} & \Gamma(M;X) \\ {\scriptstyle \phi^*}\downarrow & & {\scriptstyle \phi^*}\downarrow \\ C(M;X) & \xrightarrow{\gamma} & \Gamma(M;X) \end{array}$$

This is straightforward from the definition of γ, and gives the next proposition.

Proposition 5.3. *The map induced by γ gives a homotopy equivalence for connected CW-complexes X:*

$$ESO(n) \underset{SO(n)}{\times} C(V_{n,k}X) \to ESO(n) \underset{SO(n)}{\times} \Gamma(V_{n,k};X)\,.$$

Next, consider the standard metric on $S^n \subseteq \mathbb{R}^{n+1}$, and notice that $TS^n \oplus \mathbb{R}$ is trivializable $SO(n+1)$-equivariantly. The induced map $\gamma : C(S^n;\Sigma X) \to \Gamma(S^n;\Sigma X)$ is $SO(n+1)$-equivariant as well as a homotopy equivalence. For more examples, see [9].

Corollary 5.4. *For any connected CW-complex X, there is a homotopy equivalence*

$$E(\eta;\Sigma X) \to ESO(n) \underset{SO(n)}{\times} \Lambda^n \Sigma^{n+1} X$$

where η is obtained from the sphere bundle

$$S^n \to BSO(n) \xrightarrow{\eta} BSO(n+1)\,.$$

6. Homological calculations at the prime 2

In this section, assume that all (co)-homology groups are taken with $\mathbb{F}_2$-coefficients. Consider the next fibration obtained from the bundle $S^n \to BSO(n) \xrightarrow{\eta} BSO(n+1)$ to obtain the fibration

$$\Lambda^n S^{n+q} \to E(\eta; S^q) \to BSO(n+1).$$

Recall the suspension $E : S^{n+1} \to \Omega S^{n+q+1}$ together with the map of fibrations

$$\begin{array}{ccc} \Omega^n S^{n+q} & \xrightarrow{\Omega^n E} & \Omega^{n+1} S^{n+q+1} \\ \downarrow & & \downarrow \\ \Lambda^n S^{n+q} & \xrightarrow{\Lambda^n E} & \Lambda^n \Omega S^{n+q+1} \\ \downarrow & & \downarrow \\ S^{n+q} & \xrightarrow{E} & \Omega S^{n+q+1}. \end{array}$$

Since $\Lambda^n \Omega S^{n+q+1}$ is homotopy equivalent to $\Omega S^{n+q+1} \times \Omega^{n+1} S^{n+q+1}$ and $\Omega^n E$ induces a monomorphism in homology, the next lemma follows.

Lemma 6.1. *There is an isomorphism*

$$H^* \Lambda^n S^{n+q} \cong H^* S^{n+q} \otimes H^* \Omega^n S^{n+q}\,.$$

Next, notice that there is a factorization $C(\mathbb{R}^n; X) \to E(\eta; X) \to C(\mathbb{R}^\infty; X)$ where the composite is homotopic to the stabilization map. Thus the composite $\Omega^n S^{n+q} \to \Lambda^n S^{n+q} \to E(\eta; S^q) \to \Omega^\infty S^{\infty+q}$ induces a monomorphism in mod-2 homology.

Recall the homology of $\Omega^n S^{n+q}$. If $q > 0$, there is an isomorphism of Hopf algebras

$$H_*\Omega^n S^{n+q} \to \mathbb{F}_2[Q_I(x_q)]$$

where x_q is of degree q and $Q_I = Q_{i_1} \cdots Q_{i_{nk}}$ satisfies $0 < i_1 \leq i_2 \leq \cdots i_k \leq n-1$ with degree $(Q_i x) = i + 2$ (degree (x)). Write $Q_I x_q^*$ for the dual basis element in $H^*\Omega^n S^{n+q}$ which is dual in the standard monomial basis. Thus there is an isomorphism of algebras

$$H^*\Omega^n S^{n+q} \cong H^* S^q \otimes B_{n.q}$$

where $B_{n.q}$ is the algebra kernel of

$$H^*\Omega^n S^{n+q} \to H^* S^q$$

and $B_{n.q}$ is an exterior algebra.

Now consider the Serre spectral sequence in cohomology for the fibration $\Lambda^n S^{n+q} \to E(\eta; S^q) \to BSO(n+1)$ to get

$$E_2 \cong H^*BSO(n+1) \otimes H^*\Lambda^n S^{n+q}$$

and thus

$$E_2 \cong H^*BSO(n+1) \otimes H^*S^{n+q} \otimes H^*S^q \otimes B_{n.q}$$

as an algebra (for $q \geq 1$).

Write x_{n+q} for the fundamental class in $H^{n+q}S^{n+q}$. The following lemma describes all of the differentials in this spectral sequence.

Lemma 6.2. 1.
2. *$B_{n.q}$ consists of infinite cycles*
3. *$d_{n+1}(x_{n+q}) = x_q \cdot w_{n+1}$ where w_{n+1} is the $(n+1)$-st Stiefel-Whitney class in $H^*BSO(n+1)$, and*
4. *$x_q \cdot x_{n+q}$ is an infinite cycle.*

Before proving the lemma, consider the cohomology of $E(\eta; S^q)$ implied by these results. Notice that E_2 as a module over $H^*BSO(n+1)$ splits as a sum $A_1 \oplus A_2 \oplus A_3 \oplus A_4$ where

$A_1 = H^*BSO(n+1) \otimes B_{n.q}$,
$A_2 = H^*BSO(n+1) \otimes x_q B_{n.q}$,
$A_3 = H^*BSO(n+1) \otimes x_{q+n} B_{n.q}$, and
$A_4 = H^*BSO(n+1) \otimes x_q \cdot x_{q+n} B_{n.q}$.

Furthermore, $A_1 \oplus A_4$ consists of infinite cycles. The boundary d_n restricts to a monomorphism $d_n : A_3 \to A_4$ with cokernel $A_4/d_n A_3$. But $A_4/d_n A_3$ is isomorphic to $H^*BSO(n) \otimes x_q B_{n.q}$. Since all other elements in E_{n+1} are infinite cycles, the next theorem follows.

Theorem 6.3. *There is an isomorphism of $H^*BSO(n+1)$-modules*

$$H^*E(\eta; S^q) \cong [H^*BSO(n+1) \otimes (B_{n.q} \oplus x_q x_{q+n} B_{n.q})] \oplus [H^*BSO(n) \otimes x_q B_{n.q}] .$$

The next step is to compare the answers given in 6.3 to the filtrations of the homology of $E(\eta; S^q)$. First, notice that the k-th filtration of $E(\eta; S^q)$, E_k is homeomorphic to $ESO(n+1) \times_{SO(n+1)} F_k$ where F_k is the k-th filtration of $C(S^n; S^q)$. The next statement follows.

Lemma 6.4. *There is a map of fibrations where i is the natural inclusion:*

$$\begin{array}{ccccc} F_{k-1} & \longrightarrow & F_k & \longrightarrow & C(S^n; S^q) \\ \downarrow & & \downarrow & & \downarrow \\ E_{k-1} & \longrightarrow & E_k & \longrightarrow & E(\eta; S^q) \\ \downarrow & & \downarrow & & \downarrow \\ BSO(n+1) & \longrightarrow & BSO(n+1) & \longrightarrow & BSO(n+1) \end{array}$$

Recall that the cofibre of the natural inclusion $i : F_{k-1} \to F_k$ is $D_k(\eta; S^q)$. Furthermore, there is a stable map from F_k to F_{k-1} stably splitting i. The next two theorems follow from naturality of the Serre spectral sequence together with Proposition 2.2, and Lemma 6.4.

Theorem 6.5. 1. *If $k \equiv 0(2)$, there is an isomorphism of $H^*BSO(n+1)$-modules*

$$H^*(E_k, E_{k-1}) \cong H^*BSO(n+1) \otimes H^*(F_k, F_{k-1}) .$$

2. *If $k \equiv 1(2)$, there is an isomorphism of $H^*BSO(n+1)$-modules*

$$H^*(F_k, F_{k-1}) \cong H^*BSO(n) \otimes H^*(F_kC(\mathbb{R}^n; S^q), F_{k-1}C(\mathbb{R}^n; S^q)) .$$

Theorem 6.6. 1. *If $k \equiv 0(2)$, there is an algebra extension*

$$1 \to H^*(BSO(n+1); \mathbb{F}_2) \to H^*(E(\eta; k)/\Sigma_k; \mathbb{F}_2) \to H^*(F(S^n, k)/\Sigma_k; \mathbb{F}_2) \to 1 .$$

2. *If $k \equiv 1(2)$, there is an algebra extension*

$$1 \to H^*(BSO(n); \mathbb{F}_2) \to H^*(E(\eta; k)/\Sigma_k; \mathbb{F}_2) \to H^*(F(\mathbb{R}^n, k)/\Sigma_k; \mathbb{F}_2) \to 1 .$$

Proof. Combine 2.1 and 5.5.

To finish this section, a proof of Lemma 6.2 is given. Consider the fibration $S^n \to BSO(n) \xrightarrow{\eta} BSO(n+1)$ together with the trivial fibration $S^q \to S^q \to *$ to get a fibration $S^n \times S^q \to BSO(n) \times S^q \to BSO(n+1)$. As $S^n \times S^q = F(S^n, 1) \times_{\Sigma_1} S^q$ and $E(\eta, 1) \times_{\Sigma_1} S^q = BSO(n) \times S^q$, there is a map of fibrations given by the natural maps as follows:

$$\begin{array}{ccc} S^n \times S^q & \longrightarrow & C(S^n; S^q) \\ \downarrow & & \downarrow \\ BSO(n) \times S^q & \longrightarrow & E(\eta; S^q) \\ \downarrow & & \downarrow \\ BSO(n+1) & \longrightarrow & BSO(n+1) \end{array}$$

Furthermore, there is the map $\gamma : C(S^n; S^q) \to \Gamma(S^n; S^q)$. If $y_0 \in S^n$ is any point, evaluate a section at y_0, and obtain a map $\Gamma(S^n; S^q) \to D_{y_0}(S^n)/\partial D_{y_0}(S^n) \wedge S^q \cong S^n \wedge S^q$. On the other hand, consider the natural map

$$S^n \times S^q = F(S^n, 1) \times_{\Sigma_1} S^q \to C(S^n; S^q) .$$

Furthermore, the composition of these maps

$$S^n \times S^q \xrightarrow{j} C(S^n; S^q) \xrightarrow{\gamma} \Gamma(S^n; S^q) \xrightarrow{w} S^n \wedge S^q ,$$

gives the natural collapse map. Hence $d_{n+1}(x_{n+q}) = x_q \cdot w_{n+1}$ for the cohomology Serre spectral sequence of the left-hand side for the above morphism of fibrations. Thus $d_{n+q}(x_{n+q}) = x_q w_{n+1}$ in the right-hand side. This finishes the proof of 6.2. □

To finish the proof of Theorem 2.4, the cohomology of $F(S^n, k)/\Sigma_k$ is required. That is the subject of the next section.

7. On the cohomology of $F(S^n, k)/\Sigma_k$, and Theorem 2.4

In this section all (co)homology groups are taken with $\mathbb{F}_2$-coefficients. It was shown in [8] that there are isomorphisms of vector spaces given by

$$H_q(F(S^n, k)/\Sigma_k) \cong H_q F(\mathbb{R}^n, k)/\Sigma_k \oplus H_{q-n} F(\mathbb{R}^n, k-1)/\Sigma_{k-1} .$$

Recall that the homology groups of $F(\mathbb{R}^n, k)/\Sigma_k$ are listed in [11] and that if $k \equiv 1(2)$, the natural inclusion $F(\mathbb{R}^n, k-1)/\Sigma_{k-1} \to F(\mathbb{R}^n, k)/\Sigma_k$ gives an isomorphism in mod-2 homology.

Thus there are isomorphisms of vector spaces

$$H_q F(S^2, k)/\Sigma_k \cong H_q F(\mathbb{R}^2, k)/\Sigma_k \oplus H_{q-n} F(\mathbb{R}^2, k-1)/\Sigma_{k-1} .$$

Since $F(\mathbb{R}^2, k)/\Sigma_k$ is a $K(\pi, 1)$ where π is Artin's k-stranded braid group B_k. Theorem 2.4 follows from these remarks together with Theorem 6.6 as there are isomorphisms

$$H_q(F(S^2, k)/\Sigma_k) \to H_q B_k \oplus H_{q-2} B_{k-1}.$$

Some further consequences are listed next. From Fadell and Neuwirth [20], there is a bundle $F(\mathbb{R}^n, k-1) \to F(S^n, k) \xrightarrow{p_1} S^n$ where p_1 denotes first coordinate projection. Thus there is a bundle

$$F(\mathbb{R}^n, k-1)/\Sigma_{k-1} \to F(S^n, k)/\Sigma_{k-1} \to S^n.$$

Furthermore, the natural inclusion of S^n in $\mathbb{R}^{n+1}$ gives a commutative diagram where f classifies the cover $F(\mathbb{R}^n, k-1) \to F(\mathbb{R}^n, k-1)/\Sigma_{k-1}$.

$$\begin{array}{ccc} F(\mathbb{R}^n, k-1)/\Sigma_{k-1} & \longrightarrow & F(S^n, k)/\Sigma_{k-1} \\ {\scriptstyle f}\downarrow & & \downarrow \\ B\Sigma_{k-1} = F(\mathbb{R}^\infty, k)/\Sigma_{k-1} & \longrightarrow & B\Sigma_{k-1} \end{array}$$

Since f_* is a monomorphism in mod-2 homology [11], the next lemma follows at once.

Lemma 7.1. *There is a short exact sequence of algebras*

$$1 \to H^* S^n \to H^* F(S^n, k)/\Sigma_{k-1} \to H^* F(\mathbb{R}^n, k-1)/\Sigma_{k-1} \to 1 .$$

Remark 7.2. This extension of algebras is sometimes trivial, and sometimes not.

Next, consider the natural quotient map

$$q : F(S^n, k)/\Sigma_{k-1} \to F(S^n, k)/\Sigma_k.$$

Lemma 7.3. *If $k \equiv 1(2)$, then q^* is an isomorphism in mod-2 cohomology.*

Proof. If $k \equiv 1(2)$, then q^* must be a monomorphism as the transfer guarantees because Σ_{k-1} is of odd index in Σ_k.

That q^* is an epimorphism is a restatement of the splitting for the mod-2 homology of $F(S^n, k)/\Sigma_k$ given at the beginning of this section by the natural size comparison. □

Corollary 7.4. *If $k \equiv 1(2)$, there is a short exact sequence of algebras*

$$1 \to H^*S^n \to H^*F(S^n, k)/\Sigma_k \to H^*F(\mathbb{R}^n, k-1)/\Sigma_{k-1} \to 1$$

The next remark which is well-known [20], but is recalled for the convenience of the reader. Let $Q_3 = \{\infty \cup \pm 1\}$ denote a fixed subset of S^2 having cardinality 3.

Lemma 7.5. *There is an equivalence of bundles given as follows where $k \geq 3$ and θ is $SO(3)$-equivariant:*

$$\begin{array}{ccc}
F(S^2 - Q_3, k-3) & \xrightarrow[1]{} & F(S^2 - Q_3, k-3) \\
\downarrow & & \downarrow \\
SO(3) \times F(S^2 - Q_3, k-3) & \longrightarrow & F(S^2, k) \\
\downarrow & & \downarrow \\
SO(3) & \longrightarrow & SO(3)
\end{array}$$

Furthermore, the homotopy orbit space $ESO(3) \times_{SO(3)} F(S^2 - Q_3)$ is contractible.

Proof. Recall that $Q_3 = \{\infty \cup \pm 1\}$. The equivalence θ is defined by

$$\theta(g, (x_1, \ldots, x_{k-3}) = (g(\infty), g(1), g(-1), g(x_1), \ldots, g(x_{k-3}))\ .$$

Notice that θ satisfies the following properties:

1. The map θ is $SO(3)$-equivariant with the natural action on the target, and action specified on the source by the diagonal action where $SO(3)$ acts naturally on itself via left multiplication, and trivially on $F(S^2 - Q_3, k-3)$.
2. The map $g : SO(3) \to F(S^2, 3)$ is a homotopy equivalence.
3. The map θ induces a map of fibrations which is a homotopy equivalence on the base, and the identity on the fibre, and is thus an equivalence.
4. In case $k = 3$, $ESO(3) \times_{SO(3)} F(S^2 - Q_3)$ as $ESO(3) \times_{SO(3)} SO(3)$ is contractible.

The lemma follows.

More is true, and is well-known. The natural action of $PGL(2, \mathbb{C})$ on $\mathbb{C}P^1$ extended to a diagonal action on $F(\mathbb{C}P^1, k)$ gives rise to a homeomorphism

$$PGL(2, \mathbb{C}) \times F(S^2 - Q_3, k-3) \to F(S^2, k).$$

□

8. Odd primary calculations for $\Lambda^2 S^n$

Throughout this section, all (co)-homology groups are taken with $\mathbb{F}_p$-coefficients for p an odd prime. The cohomology groups $H^*(\Gamma^n; \mathbb{F}_p)$ will be obtained from the computations in this section.

Unfortunately, the answers are not clean. The first step is to find the cohomology of $C(S^2; S^q)$ which is homotopy equivalent to $\Lambda^2 S^{q+2}$.

Since an odd dimensional sphere localized at p is an H-space, it follows that $\Lambda^n S^{2q+1}$ is homotopy equivalent (at p) to $S^{2q+1} \times \Omega^n S^{2q+1}$.

Lemma 8.1. *If $n < 2q+1$, and S^{2q+1} is localized at an odd prime, then there is a homotopy equivalence*

$$S^{2q+1} \times \Omega^n S^{2q+1} \to \Lambda^n S^{2q+1} .$$

The bulk of the work is concerned with $\Lambda^2 S^{2q+2}$. Notice that after localization at p, there is a fibration

$$\Omega S^{2q+1} \times \Omega^2 S^{4q+3} \to \Lambda^2 S^{2q+2} \to S^{2q+2} .$$

An inspection of the Serre spectral sequence in mod-p cohomology, gives that the elements of $H^*\Omega S^{2q+1}$ are infinite cycles.

Consider the map of fibrations where E is the stabilization map:

$$\begin{array}{ccc} \Lambda^2 S^{2q+2} & \xrightarrow{\Lambda^2 E} & \Lambda^2 \Omega^\infty \Sigma^\infty S^{2q+2} \\ \downarrow & & \downarrow \\ S^{2q+2} & \xrightarrow{E} & \Omega^\infty \Sigma^\infty S^{2q+2} \end{array}$$

Since the right-hand fibration is trivial, the statement concerning $H^*\Omega S^{2q+1}$ follows at once.

To continue, the cohomology algebra $H^*\Omega^2 S^{2q+1}$ is required as a Hopf algebra over the Steenrod algebra [11]. There is an isomorphism of Hopf algebras

$$H_*\Omega^2 S^{2q+1} \to \Lambda[y_0, y_1, \dots] \otimes \mathbb{F}_p[x_1, x_2, \dots]$$

where

- **(i):** x_k, and y_k are primitive elements of degree $2p^k q - 2$ for $k \geq 1$, and $2p^k q - 1$ for $q \geq 0$ respectively,
- **(ii):** $\beta y_k = x_k$ and $P^1_* x_{k+1} = -x_k^p$ for $k \geq 1$, and
- **(iii):** $P_*^{p^j} x_k = 0$ for $j > 0$.

Thus a basis for the module of primitives which are annihilated by every Steenrod operation (in homology) is given by

$$\{y_0, x_1^{p^j} \mid j \geq 0\} .$$

Using the above basis of monomials, let $\lambda_j, j \geq 0$, denote the dual basis element in $H^*\Omega^2 S^{2q+3}$ which is dual to $x_1^{p^j}$. Write ϵ for the dual of y_0, γ_j for the j-th generator in the divided power algebra $H^*\Omega S^{2q+1}$, and i for a generator of $H^{2q+2}S^{2q+2}$. The elements ϵ and λ_j, $j \geq 0$, give generators for $H^*\Omega^2 S^{2q+3}$ as an algebra over the Steenrod algebra.

Lemma 8.2. *In the mod-p cohomology Serre spectral sequence for the fibration*

$$\Omega^2 S^{2q+2} \to \Lambda^2 S^{2q+2} \to S^{2q+2},$$

λ_j *is an infinite cycle,* γ_j *is an infinite cycle, and* $d_{2q+1}(\epsilon) = i \cdot \gamma_1$.

The proof of this lemma uses some features of the model $C(S^2; S^{2q})$ and the proof is held in abeyance. In any case, consider the following cochain complex V_{2q} which is isomorphic as an algebra to

$$H^* S^{2q+2} \otimes H^* \Omega S^{2q+1} \otimes H^* S^{4q+1}$$

where the differential is specified by $d(\epsilon) = i \cdot \gamma_1$, $d(i) = d(\gamma_j) = 0$, and d is a derivation. As the cohomology of ΩS^{2q+1} is a divided power algebra, the next observation is immediate:

(iv): $\epsilon \cdot i \cdot \gamma_j$ is a cycle, and
(v): $d(\epsilon \cdot \gamma_j) = i \cdot \gamma_1 \cdot \gamma_j = (j+1) i \cdot \gamma_{j+1}$.

Thus

(vi): $\epsilon \cdot \gamma_j$ is a cycle if and only if $j + 1 \equiv 0(p)$.

The cohomology of the cochain complex V_{2q}, $H^* V_{2q}$, together with some of the product structure is recorded next. A basis is given by

(1): $\epsilon \cdot i \cdot \gamma_j$, $j \geq 0$
(2): $\epsilon \cdot \gamma_j$, $j \equiv -1(p)$
(3): $\gamma_j, j \geq 0$, and
(4): $i \cdot \gamma_j$, $j \equiv 0(p)$.

It follows at once that algebra generators are given by

(5): γ_{p^k}, $k \geq 0$,
(6): $\epsilon \cdot i$,
(7): $\epsilon \cdot \gamma_{p-1}$, and
(8): $i \cdot \gamma_{pj}$, $j \geq 0$.

The cochain complex V_{2t} will be used to describe the Serre spectral sequence in 8.2. Let BW_{q+1} denote the Hopf algebra kernel of $(E^2)^*$ where E^2 is the double suspension $S^{2q+1} \to \Omega^2 S^{2q+3}$. As a side remark, BW_{q+1} is the cohomology of a space by work of B. Gray [23] although this fact is not used here. Thus E_2 of the Serre spectral sequence for

$$\Omega^2 S^{2q+2} \to \Lambda^2 S^{2q+2} \to S^{2q+2}$$

is given by

$$V_{2q} \otimes BW_{2q+1} .$$

By Lemma 8.2, $E_2 = E_{2q+1}$ and E_{2q+2} is isomorphic to $H^* V_{2q} \otimes BW_{2q+1}$ where $H^* V_{2q}$ denotes the cohomology of the complex V_{2q} with the prescribed differential. But $E_{2q+2} = E_\infty$ as the base of this fibration is S^{2q+2}.

Corollary 8.3. *There is an epimorphism of algebras*

$$H^* \Lambda^2 S^{2q+2} \to BW_{2q+1}$$

with algebra kernel isomorphic to $H^* V_{2q}$ *as a vector space.*

Proof of Lemma 8.2. Having given the differentials for V_{2q}, it suffices to show that BW_{2q+1} consists of infinite cycles. Since the classes $\lambda_j, j \geq 0$, are generators for BW_{2q+1} as an algebra over the Steenrod algebra, it suffices to show that the λ_j are infinite cycles. But notice that for fixed k and q sufficiently large, the λ_j are forced to be infinite cycles by a degree check which is omitted. This suffices. □

9. Odd primary cohomology of Γ^n with coefficients in the sign representation

In this section, the cohomology groups $H^*(\Gamma^n; \mathbb{F}_p(\pm 1))$ are determined where $\mathbb{F}_p(\pm 1)$ denotes the field of p elements with p an odd prime, and Γ^n acts on $\mathbb{F}_p(\pm 1)$ by $\sigma(1) = (-1)^{\text{sign}(\sigma)}$ where sign(σ) is the sign of the permutation of σ in the symmetric group Σ_n. By Proposition 4.1, there are isomorphisms for $n \geq 3$ given by

$$H_j(\Gamma^n; \mathbb{F}_p(\pm 1) \cong \bar{H}_{j+(2q+1)n} D_n(\eta; S^{2q+1})$$

where η is given in section 3, and is induced by the natural inclusion of $SO(2)$ in $SO(3)$.

Recall that the mod-p cohomology of $BSO(3)$ is that of BS^3. In addition, the homology of $\Lambda^2 S^{2n+1}$, $n \geq 1$, is filtered as is that of $E(\eta; S^n)$. Write D_k for the cofibre of the inclusion of $F_{k-1}\Lambda^2 S^{2n+1}$ in $F_k \Lambda^2 S^{2n+1}$ $(= F_k C(S^2; S^{2n-1}))$.

Theorem 9.1. *There is an isomorphism of H^*BS^3-modules given by*

$$H^*E(\eta; S^{2n-1})) \to H^*BS^3 \otimes H^*\Lambda^2 S^{2n+1}.$$

The dual isomorphism in homology preserves filtration to give an isomorphism

$$F_k H_* E(\eta; S^{2n-1}) \to H_* BS^3 \otimes F_k H_* \Lambda^2 S^{2n+1}\ .$$

Let $P^{n+1}(p)$ denote the cofibre of the degree p map on S^n. These spaces are used to identify certain special cases of Theorem 9.1 in the next corollary.

Corollary 9.2. 1. *There is an isomorphism of vector spaces*

$$\bar{H}_* D_k(\eta; S^{2n-1}) \to H_* BS^3 \otimes \bar{H}_* D_k\ .$$

2. *If $k \geq 3$, there is an isomorphism*

$$H_q(\Gamma^k; \mathbb{F}_p(\pm 1)) \to \Sigma_i H_i BS^3 \otimes \bar{H}_{q-1-k(2n-1)} D_k\ .$$

3. *If $p-1 \geq k \geq 3$, then $H_j(\Gamma^k; \mathbb{F}_p(\pm 1)) = 0$.*
4. *There are isomorphisms $H_*(\Gamma^p; \mathbb{F}_p(\pm 1)) \to H_* BS^3 \otimes \bar{H}_*(P^{p-1}(p))$.*
5. *There are isomorphisms $H_*(\Gamma^{p+1}; \mathbb{F}_p(\pm 1)) \to H_* BS^3 \otimes \bar{H}_*(P^{p-1}(p) \vee P^{p+1}(p))$.*

Proof of Theorem 9.1. Consider $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ with the direct limit toplogy. Recall that $C(\mathbb{R}^\infty; X)$ is homotopy equivalent to $\Omega^\infty \Sigma^\infty X$ for connected CW-complexes X. Embed $\mathbb{CP}^\infty$ in $\mathbb{R}^\infty$ to get a sequence of natural embeddings

(1): $\mathbb{R}^2 \subset S^2 \subset \mathbb{CP}^\infty \subset \mathbb{R}^\infty$ and
(2): $C(\mathbb{R}^2; X) \subset C(S^2; X) \subset E(\eta; X) \subset C(\mathbb{R}^\infty; X)$.

Notice that the embedding in (1) of $\mathbb{R}^2$ in $\mathbb{R}^\infty$ can be chosen to be isotopic to the standard inclusion of $\mathbb{R}^2$ in $\mathbb{R}^\infty$. Thus there is a homotopy commutative diagram where the vertical arrows are equivalences:

$$\begin{array}{ccc} \mathbb{C}(\mathbb{R}^2;X) & \xrightarrow{\sigma} & C(\mathbb{R}^\infty;X) \\ \downarrow & & \downarrow \\ \Omega^2\Sigma^2 X & \xrightarrow{\Omega^2 E} & \Omega^\infty\Sigma^\infty X \end{array}$$

Similarly, there are natural embeddings

$$C(S^2;X) \subset E(\eta;X) \subset C(\mathbb{CP}^\infty;X) \subset C(\mathbb{R}^\infty;\mathbb{CP}^\infty \bar{\times} X)$$

together a homotopy commutative diagram where $A\bar{\times}B$ denotes the half-smash product $(A \times B)/A \times \{*\}$ and the bottom arrow ϕ is given by the inclusion of $S^2 \wedge X$ in $\mathbb{CP}^\infty \wedge X$ [8]:

$$\begin{array}{ccc} C(S^2;X) & \xrightarrow{\beta} & C(\mathbb{R}^\infty;\mathbb{CP}^\infty \bar{\times} X) \\ \downarrow & & \alpha\downarrow \\ S^2 \wedge X & \xrightarrow{\phi} & C(\mathbb{R}^\infty;\mathbb{CP}^\infty \wedge X) \end{array}$$

Thus ϕ gives a monomorphism in homology by inspection.

By Corollary 5.2 $C(S^2;S^n)$ is homotopy equivalent to $S^{n+2} \times \Omega^2 S^{n+2}$ when n is odd and all spaces are localized at an odd prime p, it will follow that the inclusion $C(S^2;S^n) \subset E(\eta;S^n)$ is onto in cohomology. To see this, consider the map

$$\Delta : C(S^2;S^n) \to C(\mathbb{R}^\infty;S^n) \times C(\mathbb{R}^\infty;\mathbb{CP}^\infty \wedge S^n)$$

given by the diagonal composed with $\sigma \times \alpha\beta$ where these are the natural maps given in the previous commutative diagram.

By 8.1, and the fact that $H_*\Omega^2 S^{2q+1} \to H_*\Omega^\infty S^{\infty+2q-1}$ is a monomorphism, the map Δ induces a surjection in cohomology for n odd.

Thus the Serre spectral sequence for the fibration

$$C(S^2;S^{2n-1}) \to E(\eta;S^{2n-1}) \to BSO(3)$$

collapses. To finish, notice that the assertion about the filtration of the homology of $E(\eta;S^{2n-1})$ follows by naturality of the Serre spectral sequence. □

10. The mod-p cohomology of Γ^n

Throughout this section, assume that n is at least 3 and that p is an odd prime. All (co)-homology groups are taken with $\mathbb{F}_p$-coefficients as a trivial Γ^n-module.

In the work here, the prime 3 will have a special role. Namely, Γ^3 is isomorphic to the symmetric group on 3 letters Σ_3. Thus

1. If $p > 3$, $H^i(\Gamma^3;\mathbb{F}_p)$ is (i) $\mathbb{F}_p$ for $i = 0$, and (ii) 0 for $i > 0$.
2. If $p = 3$, then there is an isomorphism $H^*(\Gamma^3;\mathbb{F}_3) \to \Lambda[u] \otimes \mathbb{F}_3[v]$ where u is an exterior generator degree 3 and v is a polynomial generator of degree 4.

This fact forces a differential in a spectral sequence below to be non-zero if $p > 3$ while the analogous differential is zero at $p = 3$. The main work here is to show that the previously mentioned differential is the only possible non-zero differential in this spectral sequence for any odd prime p. The methods for showing this result are to detect various cohomology classes by using stable Hopf invariants (explained below). Part of the complication in the answers here stems from the fact that there is a "twisted" pattern of differentials, and the relevant spectral sequence collapses at E_5. One point which arises here and which is in contrast to the computations in section 9 is that the homology with coefficients in the sign representation $\mathbb{F}_p(\pm 1)$ behaves better.

Recall that the mod-p cohomology of $BSO(3)$ is that of BS^3.

Theorem 10.1. *If $p = 3$ and $q > 0$, there are isomorphisms of H^*BS^3-modules*

$$H^*E(\eta; S^{2q}) \cong H_*BS^3 \otimes H_*\Lambda^2 S^{2q+2} .$$

The dual isomorphism in homology preserves filtration to give an isomorphism

$$F_k H_* E(\eta; S^{2q}) \cong H_*BS^3 \otimes F_k H_* \Lambda^2 S^{2q+2} .$$

Recall from sections 8, and 9 that $D_k(S^n)$ denotes cofibre of the inclusion of the $(k-1)$-st filtration of $C(S^2; S^n)$ in the k-th filtration. The homology of the spaces $D_k(S^n)$ were used in section 9 when n is odd. The case which is used in this section is $n = 2q$.

Corollary 10.2. *If $p = 3$ and $k \geq 3$, there are isomorphisms of vector spaces*

$$H_*D_k(\eta; S^{2q}) \cong H_*BS^3 \otimes H_*D_k(S^{2q}) .$$

Thus there is an isomorphism

$$H_j(\Gamma^k) \cong \Sigma_{i \geq 0} H_i BS^3 \otimes H_{j-i-2kq} D_k(S^{2q}) .$$

This corollary is one indication of why the mod-3 cohomology with trivial coefficients for the groups Γ^n is reasonably clean. The mod p homology of $D_k(S^n)$ is given in section 8. Thus, for example, the Euler-Poincaré series for the mod-3 homology of Γ^n, $3 \leq n \leq 6$, is given in the following chart.

n	$\chi H_*(\Gamma^n; \mathbb{F}_3)$
3	$(1+t^3)/(1-t)$
4	$1/(1-t)$
5	$(1+t^3)/(1-t^4)$
6	$(1+t^3+t^4+t^5)/(1-t^4)$

In the case of Γ^6, the summand of the Poincaré series corresponding to $(1 + t^3)/(1-t^4)$ gives the mod-3 homology of $K(\Sigma_3, 1)$ which is a stable 3-local summand of $K(\Gamma^6, 1)$ [2]. In the case of $K(\Gamma^4, 1)$, this last space is stably equivalent to $K(\mathbb{Z}/3\mathbb{Z}, 1)$ after localization at $p = 3$, and that this answer for $H_*(\Gamma^n; \mathbb{F}_3)$ agrees with that of [2] for $n = 3, 4, 5, 6$.

The answers for the cohomology of Γ^n for $p > 3$ and p large with respect to n are more complicated, and require more information to state. The answers depend on the cohomology of the cochain complex V_{2q} given in section 8. To describe these answers, first define 2 vector spaces A_{2q} and U_{2q} as follows where U_{2q} is defined as a subspace of the cohomology of V_{2q}:

1. A_{2q} has a single copy of $\mathbb{F}_p$ in degree $2q(k+3)$ with basis γ_{k+3} where $(k, 3) \not\equiv 0$ mod p and (a, b) is the binomial coefficient $(a+b)!/a!b!$ for $a, b \geq 0$.
2. The vector space U_{2q} is the subspace of H^*V_{2q} with basis given by
   (1): $\epsilon \cdot i \cdot \gamma_k$ where $(k, 3) \equiv 0$ mod p,
   (2): γ_{k+3} where $(k, 3) \equiv 0$ mod p,
   (3): $\epsilon \cdot \gamma_{kp-1}$ for $k \geq 1$, and

(4): $i \cdot \gamma_{kp}$ for $k \geq 0$,
where all elements are defined in section 8. Further, recall that BW_{2q+1} is the Hopf algebra kernel of $(E^2)^* : H^*\Omega^2 S^{4q+3} \to H^* S^{4q+1}$ where E^2 is the double suspension.

Theorem 10.3. *If $p \geq 5$, there is an isomorphism of vector spaces given by*

$$H^*E(\eta; S^{2q}) \to [A_{2q} \oplus (\mathbb{F}_p[u_4] \otimes U_{2q})] \otimes BW_{2q+1}.$$

As an example, there is a homomorphism $f : \mathbb{Z}/p\mathbb{Z} \to \Gamma^{p+1}$ which gives a mod-p homology isomorphism [13]. This isomorphism can be interpreted as an explanation for the presence of "all" Dyer-Lashof operations in the homology of certain other mapping class groups.

In addition, the mod-p homology of Γ^n for all n follows from Theorem 10.3. Explicit bases also follow directly from the proof below.

The main step is to work out differentials that give the above computation. This will follow from a sequence of lemmas.

Lemma 10.4. *The differentials in the Serre spectral sequence for the fibration*

$$E(\eta; X) \to BSO(3)$$

preserve filtration.

Proof. There is a morphism of fibrations

$$\begin{array}{ccc} F_kC(S^2;X) & \longrightarrow & C(S^2;X) \\ \downarrow & & \downarrow \\ F_kE(\eta;X) & \longrightarrow & E(\eta;X) \\ \downarrow & & \downarrow \\ BSO(3) & \xrightarrow{1} & BSO(3). \end{array}$$

The result follows by naturality of the Serre spectral sequence in homology. □

Lemma 10.5. *If p is any odd prime, the classes $\gamma_j, i \cdot \gamma_{pk}$, and $\epsilon \cdot \gamma_{p-1}$, $j, k \geq 0$, are infinite cycles in the Serre spectral sequence for $E(\eta, S^{2q}) \to BSO(3)$.*

Proof. This lemma is a restatement of results in Section 8. □

The remaining generators for the cohomology of $C(S^2; S^{2q})$ as an algebra over the Steenrod algebra are given by λ_j, $j \geq 1$. The elements λ_j pullback to classes which are the dual to the homology classes $x_1^{p^j}$ in the homology of $\Omega^2 S^{2q+2}$ as described in Section 8. The technology of stable Hopf invariants will be used to show that $x_1^{p^j}$ has non-trivial image in the homology of $E(\eta; S^{2q})$. This suffices.

The main result here is as follows.

Lemma 10.6. *If p is any odd prime and $j \geq 0$, then λ_j is an infinite cycle in the homology Serre spectral sequence for $E(\eta; S^{2q}) \to BSO(3)$.*

Proof. By the above remarks, it suffices to study the map

$$C(\mathbb{R}^2; S^{2q}) \subset E(\eta; S^{2q})$$

obtained from inclusions

$$\mathbb{R}^2 \subset S^2 \subset \mathbb{CP}^\infty .$$

Recall Proposition 2.3 which states that $E(\eta; X)$ stably splits as $\bigvee_{k\geq 1} D_k(\eta; X)$ for connected CW-complexes X.

The next step is to consider stable Hopf invariants

$$h_k : E(\eta; X) \to \Omega^\infty\Sigma^\infty D_k(\eta; X)$$

such that the adjoint of the total Hopf invariant

$$H : E(\eta; X) \to \Omega^\infty\Sigma^\infty(\bigvee_{k\geq 1} D_k(\eta; X))$$

is an equivalence. Standard computations give that the λ_j are in the image of the restriction map of the total Hopf invariant, and have non-trivial image in the mod-p cohomology of $C(\mathbb{R}^2, S^{2q})$. The details of this computation are omitted. The lemma follows at once. □

The differentials on the class $\epsilon \cdot i$ in $H^{6q+3}C(S^2; S^{2q})$ must be decided. By Lemma 10.6, this is the element of least degree in $H^*C(S^2; S^{2q})$ which can support a non-trivial differential. As differentials are filtration preserving by Lemma 10.5, the only possible non-trivial differential on $\epsilon \cdot i$ is given by

$$d_4(\epsilon \cdot i) = \alpha(\gamma_3 \cdot u_4)$$

where $H^*BSO(3) \cong \mathbb{F}_p[u_4]$. Notice that $\alpha \neq 0$ mod-p if and only if $p > 3$ in order to get the correct answer for $H^*(\Gamma^3; \mathbb{F}_p) \cong H^*(\Sigma_3; \mathbb{F}_p)$. Thus this differential is non-zero if and only if $p > 3$ as is forced by the isomorphism $H^*(\Gamma^3; \mathbb{F}_p) \cong H^*(\Sigma_3; \mathbb{F}_p)$. This result is stated next.

Lemma 10.7.

$$d_4(\epsilon \cdot i) = \begin{cases} 0 & \textit{if } p = 3 \textit{ and} \\ \alpha(\gamma_3 \cdot u_4) & \textit{if } p > 3 \textit{ with } \alpha \not\equiv 0 \textit{ mod } p . \end{cases}$$

Lemma 10.8. *The map*

$$i_* : H_*D_{2p}(\mathbb{R}^2; S^{2q}) \to H_*D_{2p}(\eta; S^{2q})$$

is a monomorphism in dimension $4qp + 2p - 2$.

Proof. First consider the mod-p homology of the space $D_{2p}(\eta; S^{2q})$. Recall that a basis for $H^*D_{2p}(S^2; S^{2q})$ is given in Section 8 by

(1): γ_{2p},
(2): $\gamma_{2p-3} \cdot (i\epsilon)$,
(3): λ_0, and
(4): $\beta(\lambda_0)$.

By Lemmas 10.6, and 10.7, the elements above are infinite cycles if $p = 3$. Furthermore, if $p > 3$, $d_4(\gamma_{2p-3} \cdot (\epsilon \cdot i)) = \alpha\gamma_{2p-3}(\gamma_3 \cdot u_4) = 0$. As this is the last possible differential, λ_0 and $\beta(\lambda_0)$ are in the image of

$$i^* : H^*D_{2p}(\eta; S^{2q}) \to H^*D_{2p}(\mathbb{R}^2; S^{2q})$$

and this map is onto in degree $4qp+2p-2$ by a degree count. The lemma follows. □

Proof of Theorems 10.1 and 10.3. Notice that if $p = 3$, Lemmas 10.6 through 10.8 immediately give that the Serre spectral sequence for $E(\eta; S^{2q}) \to BSO(3)$ collapses in mod-3 cohomology. Thus 10.1 follows.

Next, recall that the mod-p cohomology of $\Omega^2 S^{2q+1}$ is isomorphic to

$$H^* S^{2q-1} \otimes BW_q$$

as an algebra and the mod-p cohomology of $C(S^2; S^{2q})$ is isomorphic to

$$H^* V_{2q} \otimes BW_{2q+1} \ .$$

Thus the E_2-term of the Serre spectral sequence for $E(\eta; S^{2q}) \to BSO(3)$ is isomorphic to

$$\mathbb{F}_p[u_4] \otimes H^* V_{2q} \otimes BW_{2q+1} \ .$$

There is a d_4-differential given by

$$d_4(\epsilon \cdot i) = u_4 \cdot \gamma_3$$

with appropriate choice of u_4 as required by Lemma 10.8.

Furthermore, a vector space basis for $H^* V_{2q}$ is given as follows:

1. $(\epsilon \cdot i) \cdot \gamma_k$, $k \geq 0$
2. $\epsilon \gamma_{kp-1}$, $k \geq 1$,
3. γ_k, $k \geq 0$, and
4. $i \cdot \gamma_{kp}, k \geq 1$.

The next two formulas (where $(i,j) = (i+j)!/i!j!$) follow at once since d_4 is a derivation.

1. $d_4((\epsilon \cdot i) \cdot \gamma_k) = u_4 \cdot \gamma_3 \cdot \gamma_k$, and
2. $d_4((\epsilon \cdot i) \cdot \gamma_k) = (k,3) u_4 \cdot \gamma_{3+k}$.

Thus $(\epsilon i) \cdot \gamma_k$ is a cycle in E_4 if and only if $(k,3)$ is zero mod p.

To compute E_5, split $H^* V_{2q}$ into a direct sum as follows: Define U_1 in $H^* V_{2q}$ to be the vector space spanned by $(\epsilon \cdot i) \cdot \gamma_k$ and γ_{k+3} where $(k,3) \not\equiv 0 \bmod p$. Recall that U_{2q} is the vector spanned by

(i): $\epsilon i \gamma_k$ and γ_{k+3} where $(k,3) \equiv 0 \bmod p$,
(ii): $\epsilon i \gamma_{kp-1}$, $k \geq 1$, and
(iii): $i \gamma_{kp}$, $k \geq 1$.

Thus, by the previous paragraph E_4 is isomorphic to

$$\mathbb{F}_p[u_4] \otimes [(U_1 \oplus U_{2q}) \otimes BW_{2q+1}]$$

as a differential $\mathbb{F}_p[u_4]$-module and thus there is an isomorphism of $H^* BS^3$-modules

$$E_5 \cong (A_{2q} \otimes BW_{2q+1}) \oplus (\mathbb{F}_p[u_4] \otimes U_{2q} \oplus BW_{2q+1}) \ .$$

But E_5 consists of infinite cycles by Lemmas 10.6 through 10.8. Hence $E_5 = E_\infty$. □

References

[1] M. G. Barratt and P. J. Eccles, Γ^+*-structures. I. A free group functor for stable homotopy theory*, Topology **13** (1974), 25–45, Γ^+*-structures. II. A recognition principle for infinite loop spaces*, Topology **13** (1974), 113–126, and Γ^+*-structures. III. The stable structure of* $\Omega^\infty \Sigma^\infty A$, Topology **13** (1974), 199–207.

[2] D. Benson, and F. R. Cohen, *Mapping class groups for surfaces of low genus*, Memoirs of the A.M.S., **90** (1991), no. 443.

[3] P. Bergau and J. Mennicke, *Über topologische Abbitdungen der Brezelfläche vom Geschlecht 2*, Math. Zeit. **74** (1960), 414–435.

[4] J. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Math. Studies, **82** (1974), Princeton University Press.

[5] J. Birman and H. Hilden. *Isotopies of homeomorphisms of Riemann surfaces*, Ann. of Math., **97** (1973), 424–439.

[6] C.-F. Bödigheimer, *Stable Splitting of Mapping Spaces*, Springer-Verlag, Lecture Notes in Math., **1286** (1987), 174–187.

[7] C.-F. Bödigheimer, F. R. Cohen, and M. Peim, *Mapping class groups, and function spaces*, Mathmatica Göttingensis preprint, Heft 5 (1989).

[8] C.-F. Bödigheimer, F. R. Cohen, and L. R. Taylor, *On the homology of configuration spaces*, Topology, **28** (1989), 111-123.

[9] C.-F. Bödigheimer and I. Madsen, *Homotopy quotients of mapping spaces and their stable splitting*, Quart. J. Math. Oxford Ser.(2)**39** (1988), 401–409.

[10] A. Brownstein and R. Lee, *Cohomology of the symplectic group* $Sp(4,\mathbb{Z})$. *I. The odd torsion case*, Trans. Amer. Math. Soc. **334** (1992), 575–596, and *Cohomology of the symplectic group* $Sp(4,\mathbb{Z})$. *II. Computations at the prime* 2 Michigan Math. J. **41** (1994), 181–208.

[11] F. R. Cohen, *Homology of* C_{n+1}*-spaces* , Springer-Verlag, L. N. M. **533** (1976), 207–351.

[12] F. R. Cohen, *The unstable decomposition of* $\Omega^2\Sigma^2X$ *and its applications*, Math. Zeit. **182** (1983), 553–568.

[13] F. R. Cohen, *Artin's braid group, classical homotopy theory and sundry other curiosities*, Contemp. Math, conference proceedings on Artin's braid groups, **76** (1987),167–206.

[14] F. R. Cohen, *On the hyperelliptic mapping class groups,* $SO(3)$*, and* $Spin^c(3)$, American J. Math., **115** (1993), 389–434.

[15] F. R. Cohen, *On mapping class groups from a homotopy theoretic point of view*, Proceedings of the 1994 Barcelona Algebraic Topology Conference, Birkäuser, Progress in Mathematics, **136** (1996), 119–142.

[16] F. R. Cohen, J. P. May and L. R. Taylor, *A stable decomposition for certain spaces* CX, Proceedings of the Cambridge Philosophical Society, **84**(1978), 465–496.

[17] F. R. Cohen, and J. Pakianathan *Course Notes*, in preparation.

[18] F. R. Cohen and L. R. Taylor, *Computations of Gelfund-Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces*, Springer-Verlag, L. N. M. **657** (1978), 106–143.

[19] E. Dyer and R. Lashof, *Homology of iterated loop spaces*, Amer. J. Math., **84** (1962), 35–88.

[20] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. **10** (1963), 111–118.

[21] E. Fadell and J. Van Buskirk, *The braid groups of* E^2 *and* S^2, Duke Math. J., **29** (1963), 243–258.

[22] D. Fuks, *Cohomology of the braid group mod-2*, Funktsional Anal. i Prilozhen, **4** (1970), 62–33; English translation in Functional Analysis and its Applications, **8** (1974),143–151.

[23] B. Gray, *On the iterated suspension*, Topology, **27** (1988), 301–310.

[24] K. Iwama, *Constructing mapping spaces from Moore spaces via little cubes*, Master's Thesis, Department of Mathematics, Shinshu University, (1998).

[25] S. Kallel, *An analogue of the May-Milgram model with multiplicities*, to appear.

[26] F. Kato, *On the space realizing the intermediate stages of the Hurewicz homomorphism*, Master's Thesis, Department of Mathematics, Shinshu University, (1996).

[27] N. Kawazumi, *Homology of hyperelliptic mapping class groups for surfaces*, Topology and its Applications, **76** (1997), 205–216.

[28] R. Lee and S. Weintraub, *Cohomology of* $Sp_4(\mathbb{Z})$ *and related groups and spaces*, Topology, **24** (1985), 291–310.

[29] J. P. May, *The Geometry of Iterated Loop Spaces*, Springer-Verlag, L. N. M., **271** (1972).

[30] R. J. Milgram, *Iterated loop spaces*, Ann. of Math., **84** (1966), 386–403.

[31] G. Segal, *Configuration spaces and iterated loop spaces*, Invent. Math., **21** (1973), 213–221.

[32] J. Smith, *Simplicial group models for* $\Omega^n\Sigma^nX$, Israel J. Math., **66** (1989), 330–350.

[33] F. V. Vainshtein, *Cohomology of braid groups*, Funct. Anal. Appl., **12** (1978), 72–73.

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