Chapter 5

The Uniformization Map

In the last chapter we come to define the uniformization map: it associates to the conformal class of directed Riemann surface the similarity class of a parallel slit domain $\mathfrak{h} : \vec{\mathfrak{M}}(g) \longrightarrow \mathfrak{PS}\mathbb{C}(g)$.

We start with the definition of a map H from an auxilliary moduli space to $PS\mathbb{C}(g)$ and show that it and its inverse are bijective, equivariant and continuous. Then the main result follows: $\mathfrak{h}$ is a homeomorphism.

5.1 The map $H : \vec{M}(g) \longrightarrow PS\mathbb{C}(g)$.

In order to define the uniformization map $\mathfrak{h} : \vec{\mathfrak{m}}(g) \longrightarrow \mathfrak{PS}\mathbb{C}(g)$ we first introduce a new moduli space $\vec{M}(g)$. The elements of $\vec{M}(g)$ are equivalence classes of tupels (F,X,α,b) : F is a Riemann surface of genus g as before; X is a direction at some point $P \in F$; $\alpha = (z : a)$ is a projective class of a local parameter z and some real number $a > 0$, z is adopted to X (i.e. $z(P) = 0$, $\vec{D}z(X) = \vec{dx}$) ; $b = b_1 + ib_2$ is any complex number. Two such tuples are conformally equivalent if there is a conformal map $c : F \longrightarrow F'$ such that $c(P) = P'$, $\vec{D}c(X) = X'$ and $a\cdot(z' \circ c) = a'\cdot z$, and $b = b'$. There are two projections

$$(5.1.1)\qquad \begin{array}{ccccc} \vec{\mathfrak{m}}(g) & \longleftarrow & \vec{M}(g) & \longrightarrow & \mathbb{R}_+ \times \mathbb{C} \\ [F,X] & \longleftarrow\!\shortmid & [F,X,\alpha,b] & \longmapsto & (a\frac{z_o}{z}, b) \end{array}$$

On the left side $\frac{z_o}{z}$ means the proportionality of two directed local parameters near P , where z_o is some fixed local parameter. The projection $\vec{M}(g) \longrightarrow \mathbb{R}_+$ is thus well-defined, but depends on the choice of z_o . Both projections exhibit $\vec{M}(g)$ as a product

$$(5.1.2)\qquad \vec{M}(g) \cong \vec{\mathfrak{m}}(g) \times \mathbb{R}_+ \times \mathbb{C} \; .$$

$\vec{M}(g)$ can be regarded as the moduli space of pairs $(F,(z:w)$, where $(z:u)$ is a projective class consisting of a directed local parameter and a holomorphic function w with a dipol of the form $w(z) = \frac{1}{z}$ + regular terms as only singularity. This determines P . And X is the argument and a is the modulus of the residuum of w measured in a fixed directed parameter z_o .

The definition of the map

$$(5.1.3) \qquad H : \vec{M}(g) \longrightarrow PS\mathbb{C}(g)$$

needs some preparation. Let a conformal equivalence class $[F,x]$, a projective parameter $\alpha = (z:a)$ and a complex number $b = b_1 + ib_2$ kept fixed till (5.1.25) .

By Proposition (3.1.3) there is a unique harmonic function u on Z with a dipol singularity at P for α and b_1 .

Let $\Phi = \mathrm{grad}_u$ be the gradient flow of u , and $\mathbb{K}_o$ be the critical graph of Φ . Let v denote the harmonic conjugate of u , defined on $F - \mathbb{K}_o = F_o$ and normalized by b_2 . So far everything was uniquely determined by $[F,x,\alpha,b] \in \vec{M}(g)$.

Now we choose a branching graph $\mathfrak{L}$ over the critical graph $\mathbb{K}_o$; this amounts, by the algorithm (3.5.12) to choosing

(5.1.4) (1) a degeneracy function

$$\varepsilon_S : \vec{\mathfrak{C}}(S) \longrightarrow \vec{\mathfrak{K}}(S) , \qquad \delta_S : \overleftarrow{\mathfrak{C}}(S) \longrightarrow \overleftarrow{\mathfrak{K}}(S)$$

for each vertex S in $\mathbb{K}_o$, $S \neq P$, and

(2) a shuffle function

$$\sigma_k : \sigma_k : \overleftarrow{\mathfrak{C}}(S) \supseteq \delta_S^{-1}(K) \longrightarrow \varepsilon_{S'}^{-1}(K) \subseteq \vec{\mathfrak{C}}(S)$$

for each edge $K = (S,S')$ from S to S' in $\mathbb{K}_o$, $S' \neq P$.

The algorithm Proposition (3.5.12) completes the data by producing

(3) the degeneracy functions

$$\delta_S : \overleftarrow{\mathfrak{C}}(S) \longrightarrow \overleftarrow{\mathbb{K}}(S)$$

for each vertex S in $\mathbb{K}_o$, $S \neq P.$, and

(4) the branching functions

$$\beta_S^+ , \beta_S^- : \overleftarrow{\mathfrak{C}}(S) \longrightarrow \overleftarrow{\mathfrak{C}}(S)$$

for each vertex S in $\mathbb{K}_o$, $S \neq P$. The properties (3.5.10) (3.5.10) now hold.

These choices determine the branching complex $\overline{F} \cup \mathbb{B} = \overline{F}$. The extension of the mapping function $w = u + iv$ from F_o to $\overline{F}$ is uniquely possible, given $\overline{F}$ and u, v on F_o . Recall that the 4g edges of $\overline{F}$ come with an ordinary

(5) $B_1, B_2, \ldots, B_{4g}$

and with a pairing function of their indices

(6) $\lambda : \{1,2,\ldots,4g\} \longrightarrow \{1,2,\ldots,4g\}$

satisfying (3.6.13) .

λ is determined by the β_S^+ and β_S^- , and the ordering in part also by v .

The images $L_i = w(B_i)$ of these edges under the (extended) mapping function w are semi-infinite slits in the complex plane $\mathbb{C}$, their endpoints are denoted by S_i . λ is considered as a pairing function of the indices of the L_i . The ordering of the B_i was arranged, so that

$$(7) \qquad v(B_i) \geqq v(B_{i+1}) \quad , \qquad i=1,\dots,4g-1 \quad .$$

(N.B. v is constant on each B_i); thus

$$(8) \qquad \mathrm{Im}(S_i) \geqq \mathrm{Im}(S_{i+1}) \quad .$$

Since only edges over the same vertex S in $\mathbb{K}_o$ are paired by λ , see (3.6.12) , we have

$$(9) \qquad \mathrm{Re}(S_i) = \mathrm{Re}(S_{\lambda(i)}) \quad , \qquad i=1,\dots,4g \quad .$$

Therefore $L = (L_i,\dots,L_{4g};\lambda)$ is a configuration of slit pairs. We set

(5.1.5) $\qquad H([F,X],\alpha,b) := [L_1,\dots,L_{4g};\lambda]$

and need the following two lemmata to justify our definition.

(5.1.5) $\qquad$ <u>Lemma.</u> <u>The equivalence class</u> $\mathcal{L} = [L] = [L_1,\dots,L_{4g};\lambda]$ <u>is well-defined.</u>

Proof: We begin with two considerations. First, let over the vertex S of $\mathbb{K}_o$ the degeneracy function

(5.1.7) $\qquad \varepsilon = \varepsilon_S : \vec{\mathfrak{C}} = \vec{\mathfrak{C}}(S) \longrightarrow \vec{\mathbb{K}} = \vec{\mathbb{K}}(S)$

be given. The sets $\vec{\mathfrak{C}}$ and $\vec{\mathbb{K}}$ are cyclic, and we name their elements

(5.1.8) $\qquad C_1 < C_2 < \dots < C_{2m} < C_1$

$\qquad\qquad K_1 < K_2 < \dots < K_{m+1} < K_1$

where $m = m(S)$ is the Morse index of S . Without loss of generality assume

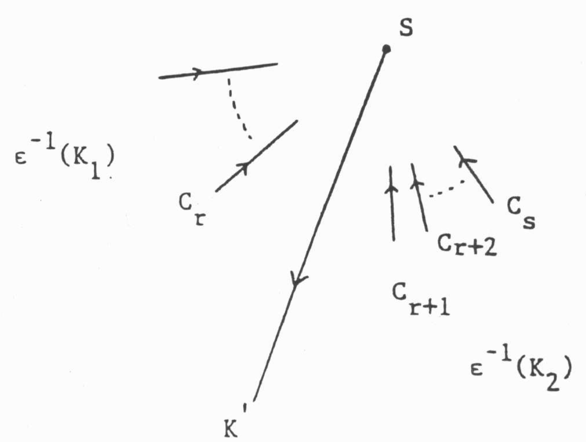
$\varepsilon^{-1}(K_1) = \{C_1,\dots,C_r\}$, and $\varepsilon^{-1}(K_2) = \{C_{r+1},\dots,C_s\}$, $1 \leqq r < s \leqq 2m$. Suppose also $s \geqq r+2$ (which need not be true). Consider the following change: redefine ε by setting

$$
\begin{aligned}
(5.1.9)\qquad \varepsilon'(C_i) &= \varepsilon(C_i) = K_1 && i = 1,\dots,r \\
\varepsilon'(C_{r+1}) &= K_1 && \\
\varepsilon'(C_i) &= \varepsilon(C_i) = K_2 && i = r+2,\dots,s \\
\varepsilon'(C_i) &= \varepsilon(C_i) && i = s+1,\dots,2m
\end{aligned}
$$

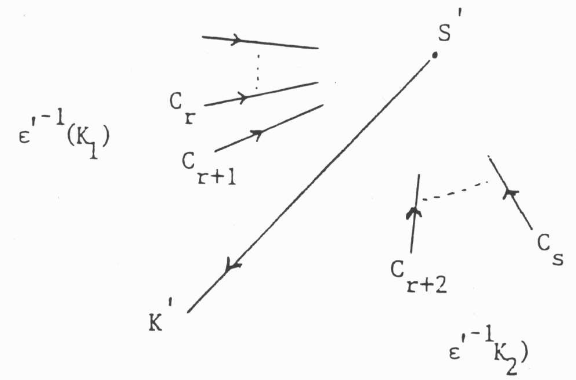
The only difference is that C_{r+1} now lies over K_1 , where formerly it lay over K_2 . For obvious reasons we call this move crossing C_{r+1} over from $\varepsilon^{-1}(K_2)$ to $\varepsilon^{-1}(K_1)$. One can perform such a crossing-over whenever $\varepsilon^{-1}(K_2)$ has at least two elements (otherwise we would violate (3.5.10) (i) with the new ε').

We analyse the effect on the algorithm (3.5.12).

(5.1.10)

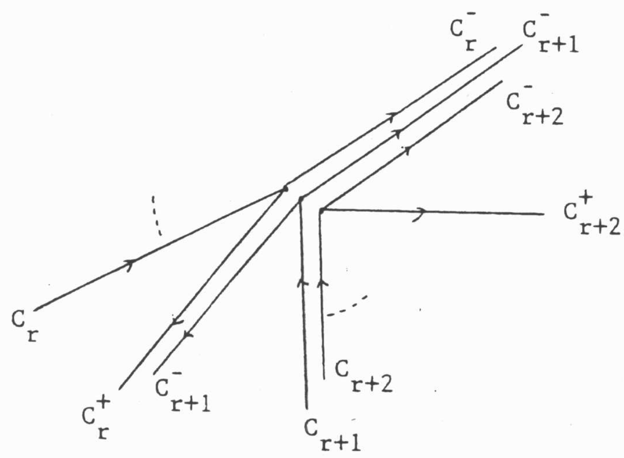


(5.1.11)



Applying the algorithm to ε gives

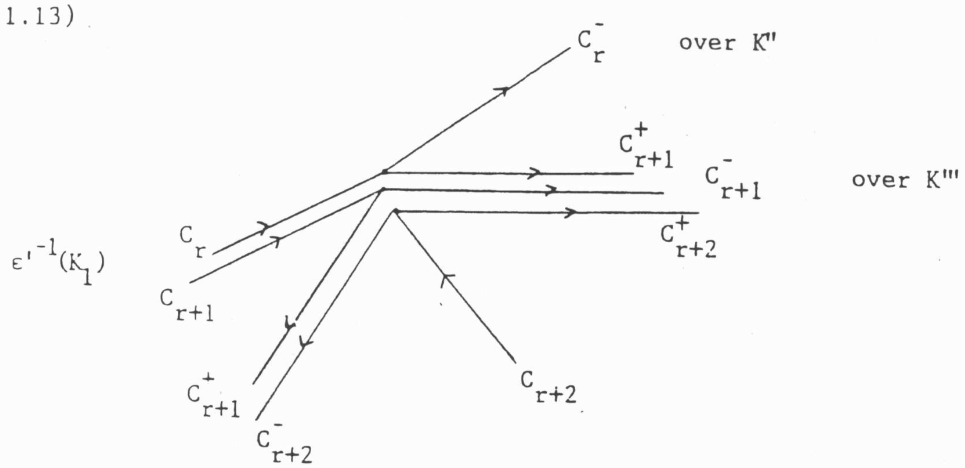
(5.1.12)



and $C_r^+ = \beta^+(C_r)$, $C_{r+1}^- = \beta^-\ (r+1)$ over the same edge K' leaving S in $\mathbb{K}_o$.

Applying the algorithm to ε' instead gives

(5.1.13)



In the branching graphs and complexes for ε resp. ε' the pairing of the double edges $B' = \{C_r^+, C_{r+2}^-\}$ and the corresponding values v', v'', v''' of v would be as follows:

(5.1.14)

for ε : B' paired to B'' ,

$v'' = v'''$

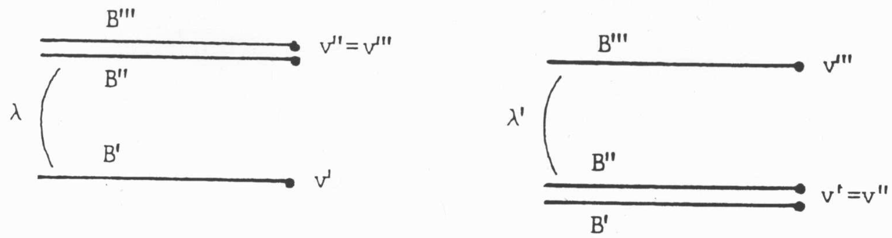
for ε': B'' paired to B''' ,

$v' = v''$

(To be correct: the values of v on the double edges depend also on the shuffle functions of all edges higher than S in $\mathbb{K}_o$; for our "local" consideration we should assume that S is a highest vertex in $\mathbb{K}_o$).

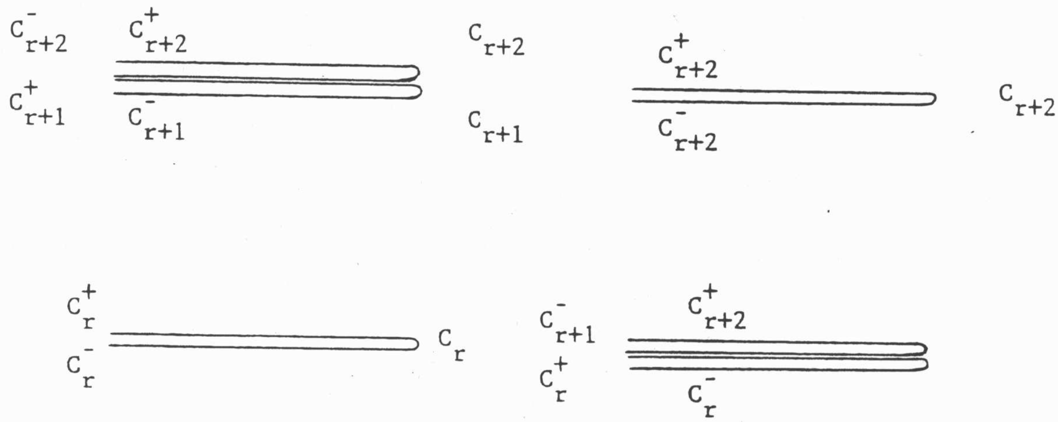
For the slits in the complex plane this means nothing but a crossing

(5.1.15) for ε : for ε' :



(The figure shows the case $v' < v'''$ and therefore a crossing-under of the slit B'' ; for $v' < v'''$ it would be a crossing-over.) Note how the pairing function changes.

(5.1.16)



The last figure is an "enlargement" of (5.1.15).

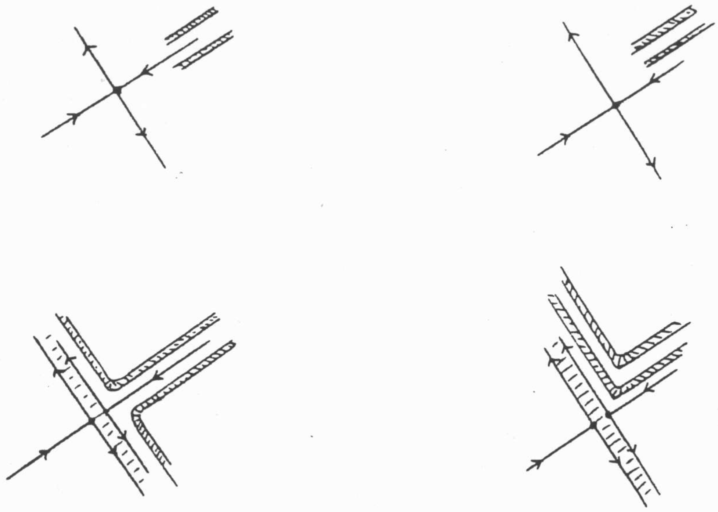
This shows: crossing some $C \in \vec{\mathfrak{C}}$, which is minimal (resp. maximal) in some $\varepsilon^{-1}(K)$, over to the cyclicly preceding (resp. succeeding) $\varepsilon^{-1}(k')$ (i.e. $K' < K$ resp. $K < K'$ in $\vec{\mathbb{K}}$) leads to equivalent configurations. Furthermore, any two given degeneracy functions ε and ε' can be transformed into each other by a sequence of such crossings.

Now secondly, let over the edge $K = (S,S')$ in $\mathbb{K}_o$ with $S' \neq P$, a shuffle function

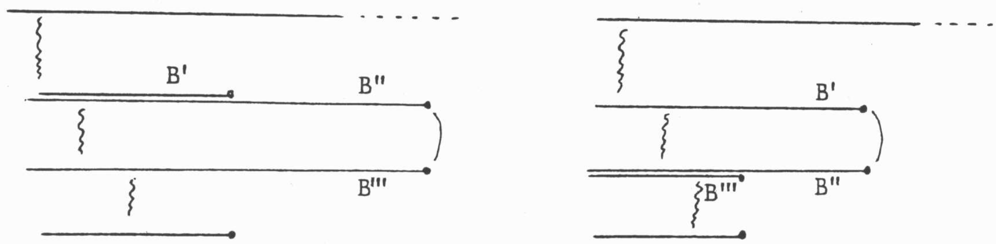
$$(5.1.17) \qquad \begin{array}{ccccc} \sigma = \sigma_K : & \delta_S^{-1}(K) & \longrightarrow & \varepsilon_{S'}^{-1}(K)^{\vee} \\ & \cap| & & \cap| \\ & \overleftarrow{\mathfrak{C}}(S) & & \overleftarrow{\mathfrak{C}}(S') \end{array}$$

be given. If the linear sets $\delta_S^{-1}(K)$ and $\varepsilon_{S'}^{-1}(K)$ have r resp. s elements, then choosing such a σ amounts to partitioning either $\delta_S^{-1}(K)$ into s successive intervals or dually $\varepsilon_{S'}^{-1}(K)$ into r . Clearly, any two partitions can be transformed into each other by a sequence of moves: crossing a minimal (resp. maximal) element of one part to the preceding (resp. succeeding) part. And by the same considerations as above for two different ε , ε' , one sees that changing a shuffle function for the branching complex results in a crossing of slits in the associated configuration.

(5.1.18) for σ : for σ' :



(5.1.19)



The combination of the two considerations, by downward induction in $\mathbb{K}_o$, starting with highest vertices, proves the lemma. ■

(5.1.20) Lemma. $L = (L_1,\dots,L_{4g};\lambda)$ is non-degenerate.

Proof: Recall from (4.2) the construction of the surface $F(L)$ associated with L as a quotient of $F' = \amalg F^k$, $k=0,\dots,4g$. We alter this construction slightly. For each slit L_k, we take two copies L_k^+, L_k^- of the extended slit (from S_k to ∞ including), regarded as right (upper) and left (lower) branch of L_k; points on these copies we denoted by $(z,k,\pm)$ if $z \in L_k$.

(5.1.21) $$F'' = \coprod_{k=0}^{4g} F^k \amalg \coprod_{k=1}^{4g} (L_k^- \amalg L_k^+)$$

we perform the gluing (4.2.3) only to the extend of (4.2.3) (1), and two other gluing-rules:

(5.1.22) (1) $(z,k) \sim (z,k+1)$

if $\mathrm{Im}(z) = \mathrm{Im}(S_k)$ and $\mathrm{Re}(z) \geq \mathrm{Re}(S_k)$,

and (2) $(z,k+) \sim (z,k,-) \sim (z,k) \sim (z,k+1)$

if $z = S_k$,

and (3) $(\infty,0,\pm) \sim (\infty,1,\pm) \sim \dots \sim (\infty,k,\pm)$.

The quotient $\overline{F}(L) = F''/(1)\dots(3)$ is homeomorphic to the branching complex $\overline{F}$ of the surface F (associated with the same branching graph which gave rise to the configuration $L = (L_1,\dots,L_{4g};\lambda)$). The homeomorphism is induced by the extended mapping function $w = u + iv : \overline{F} \longrightarrow \overline{\mathbb{C}}$ as follows. A point $z \in \overline{F}$ is mapped to $(w(z),k) \in \overline{F}(L)$ if $\mathrm{Im}(S_k) \geq v(z) \geq \mathrm{Im}(S_{k+1})$, $k=0,\dots,4g$. A point t on a double edge $B_C = B^+(C) \cup B^-(C)$ is mapped to $(t+iv(B_C),k)$ where t is the parameter of the point in $B_C \cong [-\infty,u(S)]$, $C \in \overrightarrow{\mathfrak{C}}(S)$, $v(B_C)$ is the constant value of the extended v on the double edge B_C from (3.6.5),

5.2 <u>The inverse map.</u>

To prove that H is bijective we will construct an inverse

(5.2.1) $\quad G : PS\mathbb{C}(g) \longrightarrow \vec{M}(g) \quad .$

Let $\mathcal{L} = [L_1,\dots,L_{4g};\lambda]$ be a non-degenerate configuration class. We set $G(\mathcal{L}) = [F,\chi,\alpha,b]$ with the following ingredients. F is the surface $F(\mathcal{L})$ associated to $\mathcal{L}$ in 4.2, with the conformal structure of 4.6. The basepoint is $P = \infty$, and $z = \frac{1}{\zeta}$ is a local parameter for, defined for $\zeta \notin \operatorname{supp}(\mathcal{L})$, with $z(P) = 0$. The direction χ corresponds to $-\overrightarrow{dx}$ under z. For α we take $\alpha = (z:1)$. And for $b = b_1 + ib_2$ we choose the upper right corner of $\operatorname{supp}(\mathcal{L})$, $b_i = a^+(\mathcal{L})$, $b_2 = b^+(\mathcal{L})$.

(5.2.2) <u>Proposition.</u> H <u>and</u> G <u>are inverse to each other.</u>

Proof: To see that $H \circ G = id$, recall the harmonic function $h_{\mathcal{L}}(z) = \operatorname{Re}(z) : F = F(\mathcal{L}) \longrightarrow \bar{\mathbb{R}}$ from 4.7. Then $u_{\mathcal{L}} = h_{\mathcal{L}} + b_1$ is the unique dipol function for α and b_1. Now we use the particular representative $L = (L_1,\dots,L_{4g};\lambda)$ of $\mathcal{L}$ in the following diagram.

(5.2.3)
$$\begin{tikzcd} L_k \subseteq F^{k-1}, F^k \subseteq F' \arrow[d] & F'' \supseteq L_k^+ , L_k^- \arrow[d] \\ F & \bar{F} \arrow[l] \end{tikzcd}$$

Here F' is $\amalg F^k$ from 4.2; F'' is from (5.1.21) and the projections are the various portions of the gluing - rules (4.2.3) and (5.1.22). The images of the slits L_k in F are the critical (piecewise) integral curves for the gradient flow of u. Thus and k is the future number of B_C from (3.6.11) .

There are obvious further identifications on F'' which correspond to (4.2.3) (2) and (3) ,

(4) $(z,k-1) \sim (z',\lambda(k))$

if $z \in L_k$, $z' \in L_{\lambda(k)}$ and $\mathrm{Re}(z) = \mathrm{Re}(z')$,

(5) $(z,k) \sim (z',\lambda(k)-1)$

if $z \in L_k$, $z' \in L_{\lambda(k)}$ and $\mathrm{Re}(z) = \mathrm{Re}(z')$,

(6) $(z,k-1) \sim (z,k,+)$

if $z \in L_k$,

(7) $(z,k) \sim (z,k,-)$

if $z \in L_k$.

Clearly $F''\Big/\begin{smallmatrix}(5.1.22)\\(1),\dots,(7)\end{smallmatrix} \cong F'\Big/\begin{smallmatrix}(4.2.3)\\(1),\dots,(3)\end{smallmatrix}$. But on the left hand side, the rules (4),...,(7) correspond via $F''/(1),\dots,(3) \cong \bar{F}(L)$ to identifying in the branching complex two points whenever they lie over the same point in F . Since this results in F , we see that $F(L) = F'\Big/\begin{smallmatrix}(4.2.3)\\(1),\dots,(3)\end{smallmatrix}$ is a surface of genus g ; thus L is a non-degenerate configuration. ∎

Thus we have a well-defined function

(5.1.23) $\quad H : \vec{M}(g) \longrightarrow PS\mathbb{C}(g)$

by (5.1.5) . This function is called (Hilbert) <u>uniformization map</u>.

(5.2.4) $\quad \amalg L_k^{\pm} \,/\, ^{(5.1.22)}$

is a possible choice for a branching graph $\mathfrak{C}$ over the critical graph $\mathfrak{K}_o \subseteq F$, where the precise details follow.

Over a stagnation point S in F , $S \neq P$, use all endpoints $S_{k_1},\dots,S_{k_r}$ which are glued together to become S symbolically as entering edges $\overrightarrow{\mathfrak{C}}(S)$. Note that $\{k_1,\dots,k_r\}$ must be invariant under λ . As leaving edges $\overleftarrow{\mathfrak{C}}(S)$ over S use symbolically the pairs $(L_{k_i}^+, L_{k_i}^-)$. The degeneracy functions $\varepsilon_S : \overrightarrow{\mathfrak{C}}(S) \longrightarrow \overrightarrow{\mathfrak{K}}(S)$ and $\delta_S : \overleftarrow{\mathfrak{C}}(S) \longrightarrow \overleftarrow{\mathfrak{K}}(S)$ are induced by the projections in (5.2.3) . The cyclic ordering of these sets is induced from the orientation of the local parameters in 4.6 . The branching functions β_S^+ , β_S^- are $\beta_S^+(S_{k_i}) = (L_{k_i}^+ , L_{\lambda(k_i)}^-)$ and $\beta_S^-(S_{k_i}) = (L_{\lambda(k_i)}^+, L_{k_i}^-)$. Let $K = (S,S')$ be an edge in $\mathfrak{K}_o$ from S to $S' \neq \infty$. That means the endpoints $S_{k_1},\dots,S_{k_r}$ over S and $S_{k_1'},\dots,S_{k_{r'}'}$ lie on different verticals $x = x_o$, $x = x_o'$ in $\mathbb{C}$ with $x_o < x_o'$. Since the sets of indices $\{k_1,\dots,k_r\}$ and $\{k_1',\dots,k'_{r'}\}$ are disjoint, but subsets of the ordered set $\{1,2,\dots,4g\}$, one determines a partition of the other. This is the shuffle function σ_K .

If we descend from F'' to $\overline{F}$ in (5.2.3) we obtain the branching complex $\overline{F} \cup \mathfrak{B}$ for this choice of branching data. The assertion follows now from the definition of H .

Vice versa, to see $G \circ H = \mathrm{id}$, is a repetition of the proof of Lemma 5.1 , remembering that the mapping function is not merely a homeomorphism, but a holomorphic map. ∎

5.3 Equivariance.

Consider the projection $\vec{M}(g) \longrightarrow \vec{\mathfrak{m}}(g)$ of (5.1.1) given by forgetting the projective parameter class and the integration constants. On the other side, take the orbit projection $PS\mathbb{C}(g) \longrightarrow \mathcal{PS}(g)$ induced by the action of $Sim(\mathbb{C})$ on $PS\mathbb{C}(g)$.

(5.3.1) Lemma. H and G are equivariant.

Proof: Assume $H([F',x'],\alpha',b') =: \mathcal{L}'$ and $H([F'',x''],\alpha'',b'') =: \mathcal{L}''$ are in the same $Sim(\mathbb{C})$-orbit. The element $M = \begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix} \in Sim(\mathbb{C})$ with $M \cdot \mathcal{L}' = \mathcal{L}''$ induces by (4.9.6) a conformal map $M : F(\mathcal{L}') \longrightarrow F(\mathcal{L}'')$. This map takes $\infty \in F(\mathcal{L}')$ to $\infty \in F(\mathcal{L}'')$, and the direction dx' on $F(\mathcal{L}')$ to dx'' on $F(\mathcal{L}'')$. Thus $(F(\mathcal{L}'),dx')$ and $(F(\mathcal{L}''),dx'')$ are conformally equivalent, i.e. $[F(\mathcal{L}'),dx'] = [F(\mathcal{L}''),dx'']$ in the moduli space $\vec{\mathfrak{m}}(g)$.

Let $F' \cup \mathcal{B}'$ resp. $F'' \cup \mathcal{B}''$ denote the branching complexes of F' resp. F'' . The conformal maps $w' : F' \cup \mathcal{B}' \longrightarrow \bar{\mathbb{C}}$, $w'' : F'' \cup \mathcal{B}'' \longrightarrow \bar{\mathbb{C}}$ lift to $F'(\mathcal{L}')$ resp. $F'(\mathcal{L}'')$; and then they induce conformal maps $w' : F' \longrightarrow F(\mathcal{L}')$ and $w'' : F'' \longrightarrow F(\mathcal{L}'')$. Both preserve the directions x' , x'' . From the diagram

(5.3.2)
$$\begin{array}{ccc} F' & \xrightarrow{w'} & F(\mathcal{L}') \\ & & \downarrow \\ F'' & \xrightarrow[w'']{} & F(\mathcal{L}'') \end{array}$$

we conclude that (F',x') and (F'',x'') are conformally equivalent.

Vice versa, assume $G(\mathcal{L}')$ and $G(\mathcal{L}'')$ are equivalent when projected down $\vec{M}(g) \longrightarrow \vec{\mathfrak{m}}(g)$; that means $(F(\mathcal{L}'),dx')$ and $(F(\mathcal{L}''),(dx'')$ are conformally equivalent. Let $c : F(\mathcal{L}') \longrightarrow F(\mathcal{L}'')$ be a conformal map, preserving ∞ and $dx = dx' = dx''$. Outside of the supports $\frac{1}{z} = \zeta$ is a parameter for both $(F(\mathcal{L}')$ and $F(\mathcal{L}'')$; so c must be induced by a similarity $M_c = M_c = \begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix}$ in a neighbourhood of ∞ , and hence everywhere. Thus $\mathcal{L}'$ and $\mathcal{L}''$ are in the same $\mathrm{Sim}(\mathbb{C})$-orbit . ∎

It follows, that H induces a bijection

(5.3.3) $\qquad \mathfrak{h} : \vec{\mathfrak{m}}(g) \longrightarrow \mathfrak{PS}\mathbb{C}(g)$

and G induces an inverse

$$\mathfrak{g} : \mathfrak{PS}\mathbb{C}(g) \longrightarrow \vec{\mathfrak{m}}(g) \quad .$$

In the commutative diagram

(5.3.4)
$$\begin{array}{ccc} \vec{M}(g) & \underset{G}{\overset{H}{\rightleftarrows}} & PS\mathbb{C}(g) \\ \downarrow & & \downarrow \\ \vec{\mathfrak{m}}(g) & \underset{\mathfrak{g}}{\overset{\mathfrak{h}}{\rightleftarrows}} & \mathfrak{PS}\mathbb{C}(g) \end{array}$$

the two projections are homotopy equivalences.

5.4 The continuity of H and G .

We start with the continuity of $H : \vec{M}(g) \longrightarrow PS\mathbb{C}(g)$. Varying the components α and b of a point $[F,X,\alpha,b]$ in $\vec{M}(g) \cong \vec{\mathfrak{M}}(g) \times \mathbb{R}_+ \times \mathbb{C}$ has the effect of a dilatation and translation of the corresponding configuration $H[F,X,\alpha,b] = L = [L_1,\dots,L_{4g};\lambda]$. Therefore we concentrate on the component $[F,X]$ in $\vec{\mathfrak{M}}(g)$. We need small neighbourhoods in $\vec{\mathfrak{M}}(g)$. Because of the covering (2.4.28)

$$(5.4.1) \qquad \vec{\Gamma}(g) \longrightarrow \vec{\mathfrak{T}}(g) \longrightarrow \vec{\mathfrak{M}}(g)$$

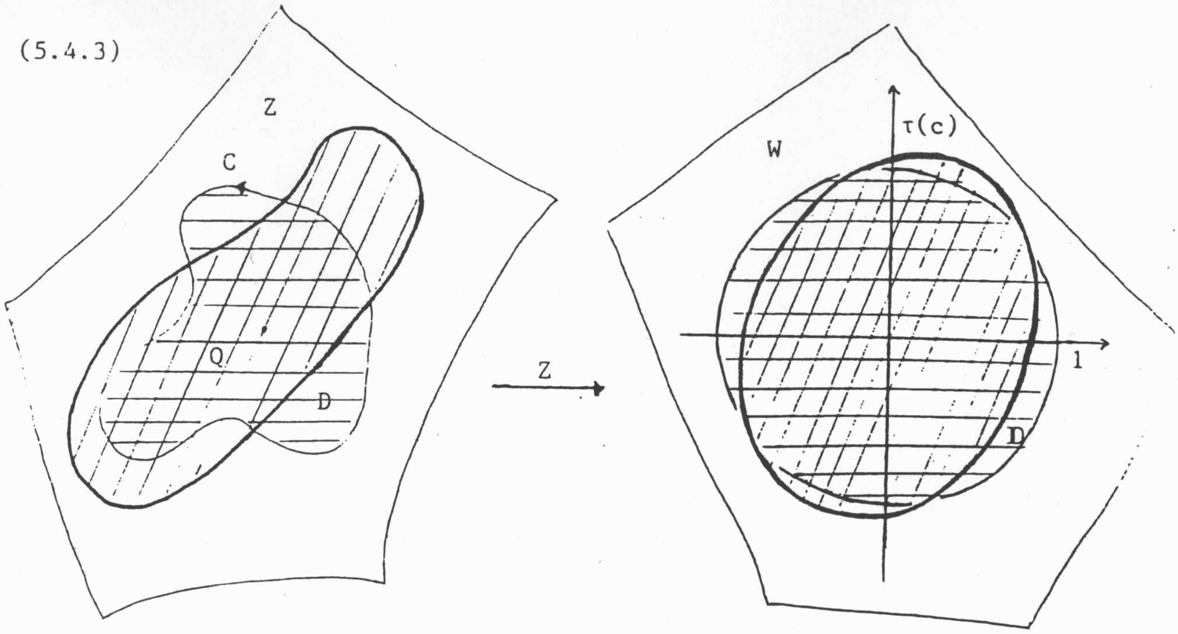
it suffices to do this in $\vec{\mathfrak{T}}(g)$. These neighbourhoods are obtained by interior Schiffer variation on a fixed Riemann surface F ; we proceed to describe this as in [Schiffer-Spencer 1954] .

Let $Q \in F$ be an arbitrary point, and $z : Z \longrightarrow \mathbb{C}$ a local parameter, $z(Q) = 0$, where $W = z(Z) \subseteq \mathbb{C}$ is assumed to contain a neighbourhood of the colose disc $\mathbb{D} \subseteq \mathbb{C}$. Set $D = z^{-1}(\mathbb{D})$, and consider the boundary curve c of D , parametrized by $z(\zeta) \in S^1 \subseteq \partial\mathbb{D}$ for $\zeta \in S^1 \cong \partial D$. For $\varepsilon \in \mathbb{C}$ consider the function τ with the presentation

$$(5.4.2) \qquad \tau(z) = z + \frac{\varepsilon}{z} .$$

in terms of the local parameter z ; it is defined and holomorphic in an annular neighbourhood of c . The image of c under τ is the ellipse $t + \varepsilon\bar{t}$ for $t = z(\zeta)$, $\zeta \in \partial D$.

(5.4.3)



If ε is small enough, $W = \mathrm{im}\, z$ contains the image of $\tau(D) = D^*$; set $D^* = z^{-1}(D^*)$. We consider now the new surface F^ε obtained by cutting on L the disc D , and gluing in the ellipse D^* ,

$$(5.4.4) \qquad F^\varepsilon = ((F - \mathrm{int}(D)) \amalg D^*) \,/\, \sim$$

where the boundary identification is given by $\zeta \sim \zeta^*$ if $z(\zeta) + \frac{\varepsilon}{z(\zeta)} = z(\zeta^*)$, for $\zeta \in \partial D$, $\zeta^* \in \partial D^*$. The new conformal structure is given by taking the old local parameters on $F - \mathrm{int}\, D$, and τ on D^* . Clearly, F and F^ε are homeomorphic, and F^0 is also conformally equivalent to F ; but for $\varepsilon \neq 0$ the conformal type changes in general. Note that the construction depends on the parameter z .

Let $Q_1,\dots,Q_n$ be different points on F , lying in disjoint charts $Z_1,\dots,Z_n$ of local parameters $z_1,\dots,z_n$. With n complex numbers $\varepsilon_1,\dots,\varepsilon_n$ one can perform independently Schiffer variations in $Z_1,\dots,Z_n$. Call the resulting new Riemann surface F^ε , $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in \mathbb{C}^n$. If one performs the variation at enough points Q_i , and if the Q_i avoid certain Weierstraß points of higher type, then the ε_i actually provide coordinates for a neighbourhood of $\langle F\rangle$ in the Teichmüller space $\mathfrak{T}(g)$.

(5.4.5) Proposition. Let F be any closed Riemann surface of genus g. Then there are points $Q_1,\dots,Q_{3g-3}$ (arbitrarily close to any given $Q'_1,\dots,Q'_{3g-3}$) on F, such that the function $\varepsilon \longrightarrow \langle F^\varepsilon\rangle$ maps some neighbourhood of 0 in C^{3g-3} homeomorphically onto a neighbourhood on $\langle F\rangle$ in $\mathfrak{T}(g)$. ■

For a proof we refer to [Gardiner 1975; 1977], [Nag 1985; 1988, p.276-278, 310]. It is immediate, that a corresponding statement also holds for pointed and directed Teichmüller spaces; one places one additional point Q_o near our basepoint P to displace P, and one further variation is centered at P itself with a parameter $\varepsilon'_o \in S^1$ to rotate the direction x .

Next we will use that the effect of a Schiffer variation on the Green's or Neumann function - and therefore on the dipol function is controllable. With the notation of [Schiffer-Spencer 1954] let Ω_{AB} be the integral of the third kind with simple poles of residues $+1,-1$ at teh points $A,B \in F$, resp., normalized by the condition that all periods are purely imaginary. For the closed surface F, one defines the Neumann function [idid., p.99] by

$$(5.4.6)\qquad N(\zeta,q,q_o) = V(\zeta,q,q_o) = V(\zeta,p_o;q,q_o)$$
$$= \mathrm{Re}(\Omega_{qq_o}(\zeta) - \Omega_{qq_o}(p_o))$$

where $p_o, q = q_o$ are fixed points on F. A dipol function at q is given by

$$(5.4.7) \qquad - u(\zeta) = \frac{\partial N(\zeta,q,q_o)}{\partial x}$$

for the local parameter $z = x + iy$ at q.

We perform a single Schiffer variation at a point Q (away from the poles p_o, $q=q_o$) with cut circle c and parameter $\varepsilon \in \mathbb{C}$. Let $\Omega^{\varepsilon}_{AB}$, V^{ε}, N^{ε} denote the corresponding functions on F^{ε}. Then, by [ibid., p.311, 312]

$$(5.4.8) \qquad V^{\varepsilon}(\zeta,p_o;q,q_o) - V^{\varepsilon}(\zeta,p_o;q,q_o) =$$

$$= \mathrm{Re}\left[\frac{1}{2\pi i}\int_c \Omega_{qq_o}(t)d\Omega^{\varepsilon}_{\zeta p_o}(t)\right]$$

$$= \mathrm{Re}\left[-\frac{2\varepsilon}{\pi i}\int \frac{1}{t}\frac{\partial}{\partial t} V(t,t_o;\zeta,p_o)\frac{\partial}{\partial t} V(t,t_o;q,q_o)dt\right] + O(\varepsilon^2)$$

where t_o serves a normalization purpose. It follows from (5.4.7), that the dipol function u^{ε} and its harmonic conjugate v^{ε} also varies continuously with the parameter ε; since u^{ε} is harmonic, the same is true for the gradient flow Φ^{ε} under a Schiffer variation. If the conformal structure is varied simultaneously at several points Q_i with variational parameters ε_i as in (5.4.5), the effects ad up for u^{ε}, $\varepsilon = (\varepsilon_1,\dots,\varepsilon_{6g-3})$,

$$(5.4.9) \qquad u^{\varepsilon} - u = \mathrm{Re}\left[-\frac{2}{\pi i}\Sigma\varepsilon_i C_i(\zeta)\right] + O(\bar{\varepsilon}^2)$$

where $C_i(\zeta)$ stands for $-\frac{\partial}{\partial x}$ of the integrand in (5.4.8) and $\bar{\varepsilon} = \Sigma\varepsilon_i$. Similarly for the gradient flow Φ^{ε}, and the harmonic conjugate v^{ε}.

After these preparations we can now prove the continuity of H.

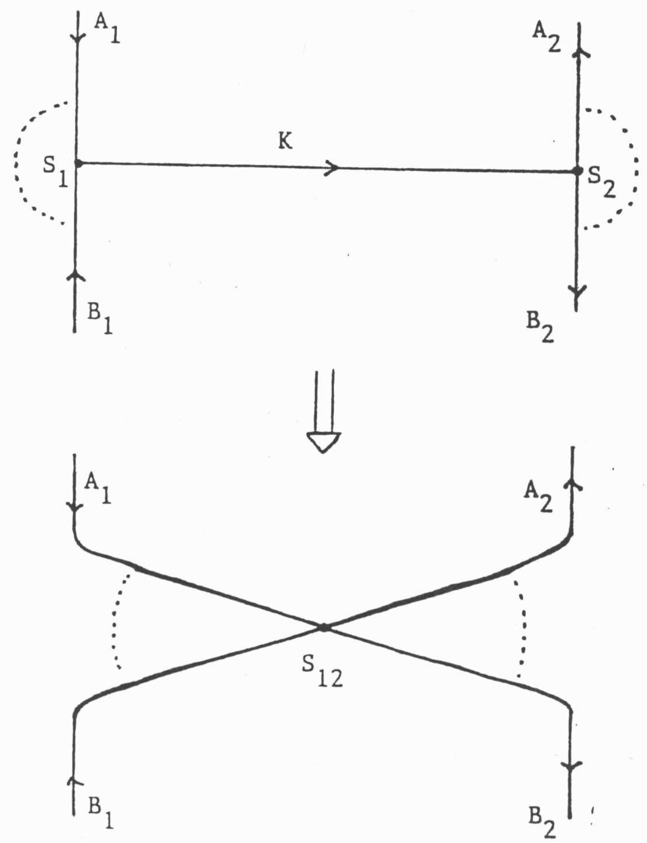
(5.4.10) <u>Lemma.</u> H <u>is continuous.</u>

Proof: Fix a point $\langle F,x\rangle$ in $\vec{\mathfrak{T}}(g)$, and consider a (multiple) Schiffer variation $\varepsilon \longrightarrow \langle F^{\varepsilon},x^{\varepsilon}\rangle$ such that $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_{3g-3},\varepsilon_o,\varepsilon_o')$ are coordinates for a neighbourhood of $\langle F,x\rangle$. Denote $H[F,x] = [L] = [L_1,\ldots,L_{4g};\lambda]$. Let $t \longmapsto \varepsilon(t)$ $(0 \leq t)$ be any curve of variational parameters with $\varepsilon(0) = 0$, and denote $F^{\varepsilon(t)}$ by F^t , and $H[F^t,x^t]$ by $[L^t]$. It is enough to see that $[L^t]$ converges to $[L] = [L^o]$ for $t \longrightarrow 0$.

While $\mathfrak{K}_o^t$ varies with $t \longrightarrow 0$, the following changes can occur with the critical graph.

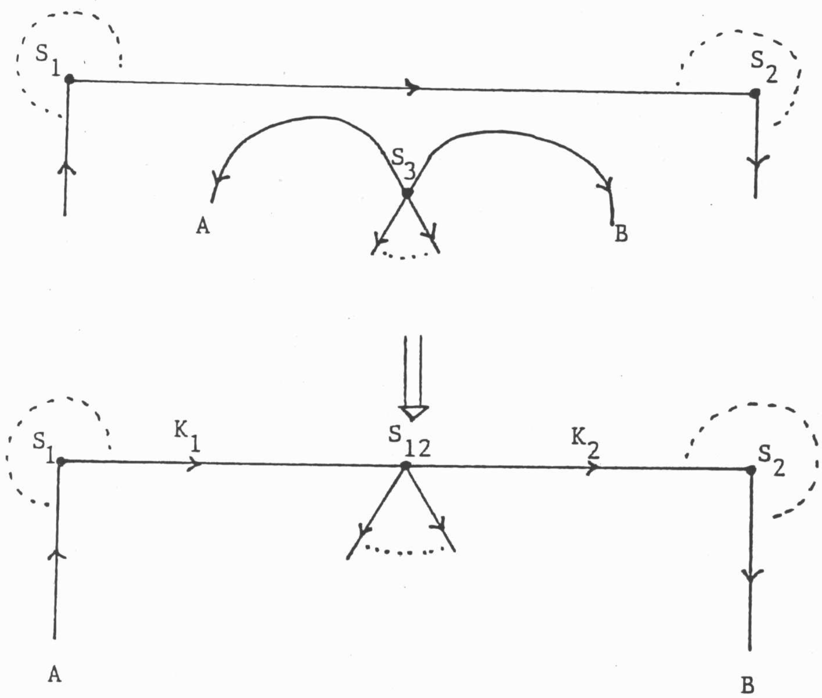
(I) <u>Two stagnation points</u> S_1 <u>and</u> S_2 <u>approach each other along a critical integral curve</u> $K = (S_1,S_2)$.

(5.4.11)



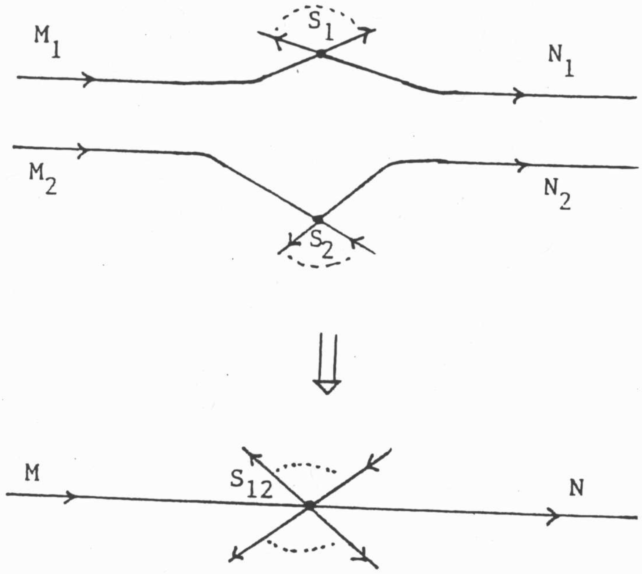
(The dotted lines indicate more integral curves entering or leaving S_1, S_2).
The edge K disappears, S_1 and S_2 become S_{12} .

(II) <u>A stagnation point</u> S_3 <u>approaches a critical integral curve</u> K <u>between to other stagnation points</u> S_1 <u>and</u> S_2 .



Here K is split into two integral curves. (The figure shows S_3 approaching K from the right.)

(III) Two stagnation points S_1 and S_2 approach each other, but not along an integral curve.



The total picture is a superposition of these elementary changes: simultanuously, at several stagnation points and stream lines. It is enough to consider only small values of $t_o \geqq t \geqq 0$, such that the following is garanted:

(5.4.13) (1) each stagnation points either stays separate from all others for all $t \geqq 0$, or stays separate for all $t > 0$ and meets one (or several) other(s) for $t = 0$ (Case I, III);

(2) each critical stream line K either stays separate from all others for all $t \geqq 0$, or disappears only for $t = 0$ (case I), or is met by a stagnation point for $t = 0$ and breaks into two critical lines (case II), or meets another critical line only for $t = 0$ (case III).

We have to show that for each stagnation point S^t there is a choice of $\varepsilon_{S^t} : \vec{\mathfrak{C}}(S^t) \longrightarrow \vec{\mathbb{K}}(S^t)$ and for each edge K^t in $\mathbb{K}_o^t$ there is a choice of a shuffling σ_{K^t} which converge for $t \longrightarrow 0$. Secondly, it must be monotone with respect to the value distribution of v^t. Thirdly, the choices must commute with each other in order to allow the superposition. Because of (5.4.15) we need only to define the limit choice for $t = 0$. For (I) we set

(5.4.14) $$\sigma_K = [\delta_1^{-1}(K), \varepsilon_2^{-1}(K)] \qquad (t>0)$$

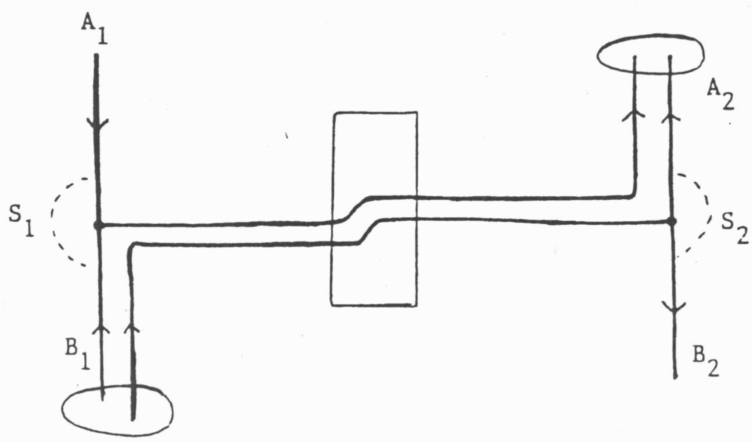
(as a linear ordering of $\delta_1^{-1}(K) \amalg \varepsilon_2^{-1}(K)$), and

$$\varepsilon_{12}^{-1}(A_1) = \varepsilon_1^{-1}(A_1), \quad \varepsilon_{12}^{-1}(B_1) = [\varepsilon_1^{-1}(B_1), \varepsilon_2^{-1}(K)], \qquad (t=0),$$

$$\delta_{12}^{-1}(A_2) = [\delta_1^{-1}(K), \delta_2^{-1}(A_2)], \quad \delta_{12}^{-1}(B_2) = \delta_2(B_2), \qquad (t=0).$$

Here we write ε_i for ε_{S_i}, and mention only what is necessary.

(5.4.15)



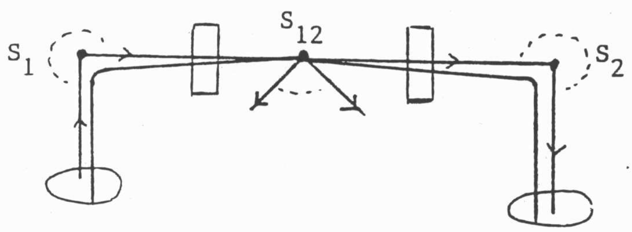
In case (II) we set for $t = 0$

$$(5.4.16) \qquad \varepsilon_{12}^{-1}(K_1) = [\delta_1^{-1}(k), \varepsilon_3^{-1}(A)] \quad ,$$

$$\delta_{12}^{-1}(K_2) = [\varepsilon_2^{-1}(K), \varepsilon_3^{-1}(B)] \quad ,$$

$$\sigma_{K_1} = [\sigma_K, \sigma_A] \quad , \qquad \text{and}$$

$$\sigma_{K_2} = [K_K, \sigma_B] \quad .$$



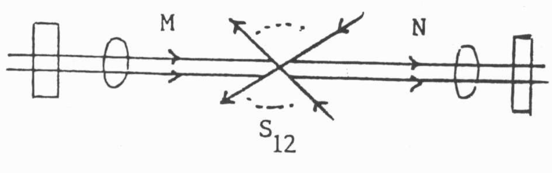
And in case (III) we set for $t = 0$

$$(5.4.17) \qquad \varepsilon_{12}^{-1}(M) = [\varepsilon_1^{-1}(M_1), \varepsilon_2^{-1}(M_2)] \quad ,$$

$$\delta_{12}^{-1}(N) = [\delta_1^{-1}(N_1), \delta_2^{-1}(N_2)] \quad ,$$

$$\sigma_M = [\sigma_{M_1}, \sigma_{M_2}] \quad , \qquad \text{and}$$

$$\sigma_N = [\sigma_{N_1}, \sigma_{N_2}]$$



From the resulting branching graphs that the branching complexes converge. Since the monotonicity (right and left sides of stream lines) are preserved, the extension of the harmonic conjugate and thus the mapping function w^t converges. It follows that $[L^t]$ converges to $[L^c]$ for $t \longrightarrow 0$. ∎

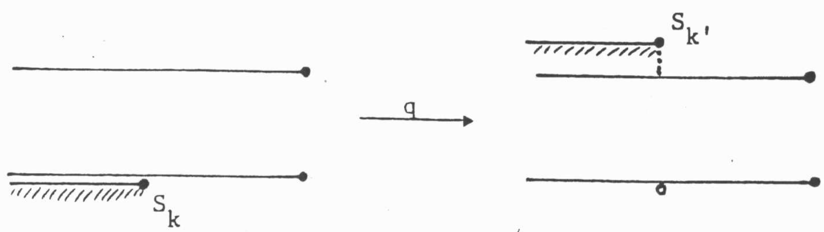
The continuity of $G : PS\mathbb{C}(g) \longrightarrow \vec{M}(g)$ is easier.

(5.4.18) Lemma. G is continuous.

Proof: Let $\mathcal{L}$ be an equivalence class in $PS\mathbb{C}(g)$, represented by the configuration $L = (L_1,\dots,L_{4g};\lambda)$. Assume, some real number $\delta > 0$ is given. We may also assume that $2\delta < \mu = \min\{\|S_i - S_j\| \mid i\neq j,\ S_i \neq S_j\}$; here $\|\ \|$ stands for the maximal-coordinate-norm in $\mathbb{C}$. Because the sets $\{Re(S_i) \mid i=1,\dots,4g\}$ and $\{Im(S_i) \mid i=1,\dots,4g\}$ are the same for any representative of $\mathcal{L}$, μ depends only on $\mathcal{L}$. But it does not vary continuously with $\mathcal{L}$; nevertheless, it suffices to concentrate on a δ-neighbourhood of $\mathcal{L}$ for $\delta < \frac{\mu}{2}$.

Recall the standard rectangulation R_{ij} $(0 \leqq i \leqq 4g\ ,\ 0 \leqq j \leqq 2g)$ of 4.12 . We consider a class $\mathcal{L}' = [L']$ with $d(\mathcal{L},\mathcal{L}') < \delta$ and compare its rectangulation R'_{ij} with R_{ij} . By the choice of δ each slit L'_k of L' differs from L_k at most by a sequence of crossings and a displacement of $S_k \longmapsto S'_k$ of a distance smaller than δ . Assume for simplicity first, only on S_k moves.

(5.4.19)



We define a quasiconformal map $q : F(L) \longrightarrow F(L')$ as follows. All rectangles R_{ij} in $F(L)$ which do not intersect the horizontal H_k correspond to rectangles $R'_{k\ell}$ of equal sides; they are mapped identically. All rectangles R_{ij} which intersect H_k and do not contain ∞ correspond to rectangles of different sides; they are mapped by an appropriate stretching map of the collection (2.3.13) - (2.3.15) . To map the remaining biangles and triangles, which contain ∞ and intersect H_k , one uses maps as (2.3.15) near their finite corner or side and extends to the rest identically. This ensures that q is the identity near ∞ .

Clearly, any L' not further away from L than δ , can be reached by a finite sequence of such simple moves. There are at most $4g$ such simple moves necessary. The composition of the corresponding maps gives a quasiconformal map $q : F(L) \longrightarrow F(L')$, conformal near ∞ and preserving the direction.

It remains to estimate the maximal dilatation. From (2.3.13) - (2.3.15) it follows that $K[q]$ on each rectangle is bounded above by some constant times δ , the constant depending on the type of map. Since there are at most $(2g+1)(4g+1)$ rectangles to be mapped non-conformally in a simple move, and since at most $4g$ simple moves are composed, $K[q]$ is of the order of δ . Thus the Teichmüller distance of $[F(L),x]$ and $[F(L'),x]$ is bounded by a constant times $d(L,L')$. This proves G to be continuous. ∎

5.5 The uniformization theorem

We come to the main result.

(5.5.1) Theorem. The Hilbert uniformization function

$$\mathfrak{h} : \vec{\mathfrak{M}}(g) \longrightarrow \mathfrak{PSC}(g)$$

is a homeomorphism.

Proof: The function H is bijective by Proposition 5.2 , continuous by Proposition 5.4 , and has the continuous inverse $H^{-1} = G$ by Proposition 5.4 . From the commutativity of the diagram (5.3.4)

(5.5.2)
$$\begin{tikzcd} \vec{M}(g) \arrow[r, "H"] \arrow[d] & PS\mathbb{C}(g) \arrow[d] \\ \vec{\mathfrak{M}}(g) \arrow[r, "\mathfrak{h}"'] & \mathfrak{PSC}(g) \end{tikzcd}$$

it follows that $\mathfrak{h}$ is bijective and continuous, with continuous inverse $\mathfrak{h}^{-1} = \mathfrak{g}$. ∎

(5.5.3) Theorem. There is a homotopy equivalence

$$\vec{\mathfrak{M}}(g) \longrightarrow PS\mathbb{C}(g) \quad .$$

Proof: An equivalence is given by composing $\mathfrak{h}$ with the normalization section $N : \mathfrak{PSC}(g) \longrightarrow PS\mathbb{C}(g)$, $\mathrm{Sim}(\mathbb{C})\mathfrak{L} \longrightarrow m(\mathfrak{L}) \cdot \mathfrak{L}$, where m is the function $PS\mathbb{C}(g) \longrightarrow \mathrm{Sim}(\mathbb{C})$ defined in the proof of Proposition 4.9.3 . ∎

(5.5.4) <u>Theorem.</u> <u>There is a homotopy equivalence</u>

$$PS\mathbb{C}(g) \simeq B\vec{\Gamma}(g) \quad .$$

Proof: The assertion combines (5.5.3) and (2.4.29) . ■

So $PS\mathbb{C}(g)$ is a convenient model for the moduli space $\vec{\mathfrak{M}}(g)$ of directed Riemann surfaces, and for the classifying space of the directed mapping class group, which is isomorphic to the relative mapping class group for surfaces with a single boundary curve.

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