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Splitting the Künneth Sequence in K-Theory *

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0. Introduction

Complex K-theory is a multiplicative cohomology theory and Atiyah [4] established an exact and natural Künneth sequence

(K) $0 \longrightarrow \tilde{K}^{\#}(X) \otimes \tilde{K}^{\#}(Y) \longrightarrow \tilde{K}^{\#}(X \wedge Y) \longrightarrow \tilde{K}^{\#}(X) * \tilde{K}^{\#}(Y) \longrightarrow 0.$

Buhštaber and Miščenko [8], Mislin [15], Puppe [19], and Anderson [1] proved that (K) splits if both respectively one of the spaces is a finite CW complex. Implicitly, the three latter papers also contain the result that (K) is always pure.

We continue the study of the Künneth sequence by applying – in addition to the multiplicative structure of the K-theory with coefficients – the Ulm theory developed for the investigation of abelian groups with elements of infinite height. Along these lines we arrive at our main result, which states that (K) is not just pure but even "transfinitely pure", i.e. Theorem 5.6. *The Künneth sequence is balanced exact for all compact X, Y.*

Since countable groups can be completely classified by means of the Ulm theory and since K-groups of compact metric spaces are countable, we obtain Theorem 5.8. *The Künneth sequence splits for all compact metric X, Y.*

This answers a question of Brown [6, p. 13, Footnote]. (N.B.: The Ext-homology theory of Brown et al. [7] is defined for compact metric spaces only.)

The paper is organised in five sections. In the Sects. 1, 3, and 4 we summarise the K-theory with coefficients and the group theory we use. In Sect. 2 we prove that the universal coefficient sequences split naturally in the group variable (in a certain sense). Section 5 contains the results about the Künneth sequence.

1. K-Theory with Coefficients

(1.1) To put coefficients $\mathbb{Z}_n$ into a cohomology theory we start with the construction of a co-Moore space $L\mathbb{Z}_n$. For $n \geq 1$ let

(1.2) $S^1 \xrightarrow{n} S^1 \xrightarrow{Pn} Cn \xrightarrow{Qn} S^2 \xrightarrow{Sn} S^2 \xrightarrow{SPn} SCn \rightarrow \ldots$

* Dedicated to Prof. A. Dold on his fiftieth birthday

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be the Puppe sequence induced by the multiplication $n: S^1 \to S^1$ (see Puppe [18, 1]). For $L\mathbb{Z}_n = Cn$ we have $\tilde{K}^0(L\mathbb{Z}_n) \cong \mathbb{Z}_n$ and $\tilde{K}^{-1}(L\mathbb{Z}_n) = 0$.

If we define $\tilde{K}^i(X;n) = \tilde{K}^{i+2}(X \wedge Cn)$ for all spaces X [by a space we always mean a compact (Hausdorff) space], this together with the induced homomorphisms and the suspension isomorphism defined in the obvious manner gives us a reduced cohomology theory, K-theory with coefficients in $\mathbb{Z}_n$ (Araki and Toda [2, 2.2], Maunder [14, 2.2], and Hilton [11, 3]).

(1.3) For any space X the sequence (1.2) gives a new Puppe sequence

(1.4) $$X \wedge S^1 \xrightarrow{1\wedge n} X \wedge S^1 \xrightarrow{1\wedge Pn} X \wedge Cn \xrightarrow{1\wedge Qn} X \wedge S^2 \xrightarrow{1\wedge Sn} X \wedge S^2 \to \dots .$$

One defines then the Bockstein homomorphism $\beta_n: \tilde{K}^i(X;n) \to \tilde{K}^{i+1}(X)$ by $\beta_n = \sigma^{-1}(1 \wedge Pn)^*$ and reduction homomorphism $\varrho_n: \tilde{K}^i(X) \to \tilde{K}^i(X;n)$ by $\varrho_n = (1 \wedge Qn)^* \sigma^2$, where σ stands for the suspension isomorphism (Araki and Toda [2, 2.2] and Maunder [14, 2.2]). From (1.4) we get at once

(1.5) **Proposition.** *The sequence*

$$\dots \leftarrow \tilde{K}^{i+1}(X) \xleftarrow{n} \tilde{K}^{i+1}(X) \xleftarrow{\beta_n} \tilde{K}^i(X;n) \xleftarrow{\varrho_n} \tilde{K}^i(X) \xleftarrow{n} \tilde{K}^i(X) \leftarrow \dots$$

is exact and natural. □

(1.6) **Corollary.**

i) $\operatorname{im} \beta_n = \tilde{K}^{i+1}(X)[n]$,
ii) $\ker \varrho_n = n\tilde{K}^i(X)$. □

[Here $\tilde{K}^{i+1}(X)[n]$ are all $x \in \tilde{K}^{i+1}(X)$ such that $nx = 0$.]

(1.7) While β_n and ϱ_n are natural transformations between the cohomology theories $\tilde{K}(\ )$ and $\tilde{K}(\ ;n)$ there are also natural transformations between the theories with different coefficient groups.

Let $n, m \geqq 1$ and (n,m) denote the greatest common divisor of n and m. The first square in the diagram

(1.8) $$\begin{array}{ccccccccc} S^1 & \xrightarrow{n} & S^1 & \xrightarrow{Pn} & Cn & \xrightarrow{Qn} & S^2 & \xrightarrow{Sn} & S^2 \to \dots \\ \downarrow{\scriptstyle \frac{n}{(n,m)}} & & \downarrow{\scriptstyle \frac{m}{(n,m)}} & & \downarrow{\scriptstyle C_{n,m}} & & \downarrow{\scriptstyle S\frac{n}{(n,m)}} & & \downarrow{\scriptstyle S\frac{m}{(n,m)}} \\ S^1 & \xrightarrow[m]{} & S^1 & \xrightarrow[Pm]{} & Cm & \xrightarrow[Qm]{} & S^2 & \xrightarrow[Sm]{} & S^2 \to \dots \end{array}$$

commutes. Therefore there is a unique map $C_{n,m}: Cn \to Cm$ such that all squares commute. We define the coefficient homomorphism $\kappa_{n,m}: \tilde{K}^i(X;m) \to \tilde{K}^i(X;n)$ by $\kappa_{n,m} = (1 \wedge C_{n,m})^*$ (compare Araki and Toda [2, 2.4 and 2.5] and Maunder [14, 2.3]).

The following proposition is an immediate consequence of the definition, the commutativity of (1.8) and the functoriality of the mapping cylinder (Puppe [18, 2.5A]).

(1.9) **Proposition.**

i) $\kappa_{n,m}$ *is natural,*

ii) $\beta_n \kappa_{n,m} = \frac{m}{(n,m)} \beta_m$,

iii) $\kappa_{n,m} \varrho_m = \varrho_n \frac{n}{(n,m)}$,

iv) $\kappa_{k,n} \kappa_{n,m} = n \frac{(k,m)}{(k,n)(n,m)} \kappa_{k,m}$, *and in particular*

v) $\kappa_{n,n} = 1$,

vi) $\kappa_{knm,nm} \kappa_{nm,m} = \kappa_{knm,m}$,

vii) $\kappa_{m,nm} \kappa_{nm,knm} = \kappa_{m,knm}$,

viii) $\kappa_{nm,knm} \kappa_{knm,km} = \kappa_{nm,m} \kappa_{m,km}$. $\square$

2. Universal Coefficient Sequences

(2.1) From diagram (1.8) and Propositions (1.5) and (1.9ii) and iii)) we get a commutative diagram of universal coefficient sequences

$$\begin{array}{ccccccccc} 0 \to & \tilde{K}^i(X)/n & \xrightarrow{\tilde{\varrho}_n} & \tilde{K}^i(X;n) & \xrightarrow{\tilde{\beta}_n} & \tilde{K}^{i+1}(X)[n] & \to 0 \\ & \uparrow{\scriptstyle \kappa'_{n,m}} & & \uparrow{\scriptstyle \kappa_{n,m}} & & \uparrow{\scriptstyle \kappa''_{n,m}} & \\ 0 \to & \tilde{K}^i(X)/m & \xrightarrow{\tilde{\varrho}_m} & \tilde{K}^i(X;m) & \xrightarrow{\tilde{\beta}_m} & \tilde{K}^{i+1}(X)[m] & \to 0 \end{array} \tag{2.2}$$

where $\tilde{\varrho}$, $\tilde{\beta}$ and κ', κ'' are induced by ϱ, β and the multiplications by $\frac{n}{(n,m)}$ in $\tilde{K}^i(X)$ resp. by $\frac{m}{(n,m)}$ in $\tilde{K}^{i+1}(X)$. Therefore we have

(2.3) **Proposition.** *For every* $n \geqq 1$ *there is a universal coefficient sequence*

$$0 \to \tilde{K}^i(X)/n \xrightarrow{\tilde{\varrho}_n} \tilde{K}^i(X;n) \xrightarrow{\tilde{\beta}_n} \tilde{K}^{i+1}(X)[n] \to 0.$$

These sequences are exact, natural in X *and compatible with the coefficient homomorphisms:*

i) $\beta_n \kappa_{n,m} = \kappa''_{n,m} \beta_m$,

ii) $\kappa_{n,m} \varrho_m = \varrho_n \kappa'_{n,m}$. $\square$

(2.4) In the proof of Theorem (2.8) about the splitting of the universal coefficient sequences we shall need several times the following result of Araki and Toda [2, Proposition 2.2, Theorem 2.3].

(2.5) **Proposition.** *For all* $n \geqq 1$, $\tilde{K}^i(X;n)$ *is a* $\mathbb{Z}_n$*-module.* $\square$

In [2] the proof is carried out for finite CW complexes, but it holds for compact spaces also; the point is that $\eta^* = 0$ for the Hopf map $\eta: S^3 \to S^2$, i.e. that

$\tilde{K}$ is a "good" cohomology theory in the sense of Hilton and Deleanu [12, 1.7, Proposition 1.9].

(2.6) *Remarks*

a) A group G [by a group we always mean an abelian group] is a $\mathbb{Z}_n$-module if and only if $nG=0$ (i.e. G is bounded by n or G has exponent n).

b) Therefore for any group G both G/n and $G[n]$ are $\mathbb{Z}_n$-modules.

c) A group G is a $\mathbb{Z}_n$-module if and only if G is a direct sum of cyclic groups whose orders divide n (Fuchs [9, Theorem 17.2]).

d) If a group G is a direct sum of cyclic groups, we call a set $\{g_i\}$, $i\in I$, of generators of G a *basis of* G if G is the direct sum of the cyclic groups $\langle g_i\rangle$, $i\in I$ (Fuchs [9, p. 78]).

(2.7) It is well known that all the universal coefficient sequences in (2.2) split unnaturally in X (Araki and Toda [2, Theorem 2.7, Corollary 2.8], Maunder [14, Theorem 2.2.1], Hilton and Deleanu [12, Corollary 3.5], and Mislin [15, Lemma 5.1]). Following the proof of Araki and Toda we now want to show that they split even naturally in n; more precisely we mean: for a fixed prime p one gets by Proposition (2.3) an inverse sequence of universal coefficient sequences

$$\begin{array}{ccccccccc}
0\to & \tilde{K}^i(X)/p & \longrightarrow & \tilde{K}^i(X;p) & \longrightarrow & \tilde{K}^{i+1}(X)[p] & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
& \vdots & & \vdots & & \vdots & \\
& \uparrow & & \uparrow & & \uparrow & \\
0\to & \tilde{K}^i(X)/p^n & \xrightarrow{\tilde{\varrho}_{p^n}} & \tilde{K}^i(X;p^n) & \xrightarrow{\tilde{\beta}_{p^n}} & \tilde{K}^{i+1}(X)[p^n] & \to 0 \\
& \uparrow{\scriptstyle \kappa'_{p^n,p^{n+1}}} & & \uparrow{\scriptstyle \kappa_{p^n,p^{n+1}}} & & \uparrow{\scriptstyle \kappa''_{p^n,p^{n+1}}} & \\
0\to & \tilde{K}^i(X)/p^{n+1} & \xrightarrow{\tilde{\varrho}_{p^{n+1}}} & \tilde{K}^i(X;p^{n+1}) & \xrightarrow{\tilde{\beta}_{p^{n+1}}} & \tilde{K}^{i+1}(X)[p^{n+1}] & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

that is, an exact sequence of inverse systems. We shall now construct a section for this sequence.

(2.8) **Theorem.** *For every* $n\geqq 1$ *there is a homomorphism* $s_{p^n}:\tilde{K}^{i+1}(X)[p^n]\to\tilde{K}^i(X:p^n)$ *such that*

i) $\tilde{\beta}_{p^n}s_{p^n}=1$,

ii) $\kappa_{p^n,p^{n+1}}s_{p^{n+1}}=s_{p^n}\kappa''_{p^n,p^{n+1}}$.

Proof. We construct the s_{p^n} recursively. Since for $n=1$ both $\tilde{K}^{i+1}(X)[p]$ and $\tilde{K}^i(X;p)$ are $\mathbb{Z}_p$-modules, i.e. vector spaces over the field $\mathbb{Z}_p$, there is a right inverse s_p for $\tilde{\beta}_p$, thus i) $\tilde{\beta}_p s_p=1$.

Assume that we have already constructed $s_p, s_{p^2},\dots,s_{p^n}$, such that

i) $\tilde{\beta}_{p^m}s_{p^m}=1$, $\quad 1\leqq m\leqq n$,

ii) $\kappa_{p^m,p^{m+1}}s_{p^{m+1}}=s_{p^m}\kappa''_{p^m,p^{m+1}}$, $\quad 1\leqq m\leqq n-1$.

$\tilde{K}^{i+1}(X)[p^{n+1}]$ is a $\mathbb{Z}_{p^{n+1}}$-module, which has a basis as explained in (2.6d). We now consider a basis element x of order $p^k, 1 \leqq k \leqq n+1$. Hence x is already in $\tilde{K}^{i+1}(X)[p^k]$ and there is an $x_{p^k} \in \tilde{K}^i(X; p^k)$ [for instance $s_{p^k}(x)$ if $k \leqq n$] such that

(1a) $\tilde{\beta}_{p^k}(x_{p^k}) = x$,

(1b) $p^k x_{p^k} = 0$,

because $\tilde{K}^i(X; p^k)$

is a $\mathbb{Z}_{p^k}$-module. For $x'_{p^{n+1}} = \kappa_{p^{n+1}, p^k}(x_{p^k})$ we get by Proposition (2.3i)

(2a) $\tilde{\beta}_{p^{n+1}}(x'_{p^{n+1}}) = x$,

(2b) $p^k x'_{p^{n+1}} = 0$.

Therefore $x'_{p^{n+1}}$ is a good preimage of x, but in order to get ii) too we have to improve it.

Consider

$$d = \kappa_{p^n, p^{n+1}}(x'_{p^{n+1}}) - s_{p^n} \kappa''_{p^n, p^{n+1}}(x)$$
$$= \kappa_{p^n, p^{n+1}}(x'_{p^{n+1}}) - s_{p^n}(px),$$

the deviation from commutativity. Since $\tilde{\beta}_{p^n}(d) = 0$, there is an $x'' \in \tilde{K}^i(X)$ such that

(3) $\tilde{\varrho}_{p^n}(x'' + p^n \tilde{K}^i(X)) = \varrho_{p^n}(x'') = d$.

If we set $x_{p^{n+1}} = x'_{p^{n+1}} - \varrho_{p^{n+1}}(x'')$, we get

(4a) $\tilde{\beta}_{p^{n+1}}(x_{p^{n+1}}) = x$,

(4c) $\kappa_{p^n, p^{n+1}}(x_{p^{n+1}}) = s_{p^n} \kappa''_{p^n, p^{n+1}}(x)$.

It remains to show that the order did not change. First we have

(5) $p^{k-1} s_{p^n} \kappa''_{p^n, p^{n+1}}(x) = p^{k-1} s_{p^n}(px) = 0$.

On the other hand

$$\kappa_{p^n, p^{n+1}}(x'_{p^{n+1}}) = \kappa_{p^n, p^{n+1}} \kappa_{p^{n+1}, p^k}(x_{p^k})$$
$$= \kappa_{p^n, p^{k-1}} \kappa_{p^{k-1}, p^k}(x_{p^k})$$

by Proposition (1.9viii); hence we also have

(6) $p^{k-1} \kappa_{p^n, p^{n+1}}(x'_{p^{n+1}}) = 0$,

because $\kappa_{p^{k-1}, p^k}(x_{p^k})$ lies in a $\mathbb{Z}_{p^{k-1}}$-module. From (5) and (6)

(7) $p^{k-1} d = 0$.

Now $\tilde{\varrho}_{p^n}$ is mono, whence (3) and (7) gives $p^{k-1}(x'' + p^n \tilde{K}^i(X)) = 0$, but that means $p^{k-1} x'' \in p^n \tilde{K}^i(X)$, and therefore $p^k x'' \in p^{n+1} \tilde{K}^i(X)$, or equivalently $p^k(x'' + p^{n+1} \tilde{K}^i(X)) = 0$. From this last statement we have

(8) $p^k \tilde{\varrho}_{p^{n+1}}(x'' + p^{n+1} \tilde{K}^i(X)) = p^k \varrho_{p^{n+1}}(x'') = 0$,

and therefore it is still true that

(4b) $p^k x_{p^{n+1}} = 0$.

By this procedure we get for every basis element x of $\tilde{K}^{i+1}(X)[p^{n+1}]$ an element $x_{p^{n+1}} \in \tilde{K}^i(X; p^{n+1})$ satisfying (4a)–(4c); putting $s_{p^{n+1}}(x) = x_{p^{n+1}}$ defines a homomorphism $s_{p^{n+1}}: \tilde{K}^{i+1}(X)[p^{n+1}] \to \tilde{K}^i(X; p^{n+1})$ such that

i) $\tilde{\beta}_{p^{n+1}} s_{p^{n+1}} = 1$,
ii) $\kappa_{p^n, p^{n+1}} s_{p^{n+1}} = s_{p^n} \kappa''_{p^n, p^{n+1}}$

hold, because this is true for every basis element according to (4a) and (4c) respectively. □

3. Multiplicative Structures

(3.1) The tensor product of vector bundles induces a cup product $\tilde{K}^i(X) \otimes \tilde{K}^j(Y) \to \tilde{K}^{i+j}(X \wedge Y)$ in the reduced K-theory (Atiyah [5, 2.6]). Thus one gets in the $\mathbb{Z}_2$-graded cohomology theory $\tilde{K}^* = \tilde{K}^0 \oplus \tilde{K}^{-1}$ a multiplication $\mu: \tilde{K}^*(X) \otimes \tilde{K}^*(Y) \to \tilde{K}^*(X \wedge Y)$.

(3.2) In [4], Atiyah has shown how to construct the Künneth sequence (K) for two finite CW complexes X, Y, with Lemmas 1 and 2 as the essential steps. But Lemma 1 is valid also for compact X, Y: one approximates X by an inverse system of finite CW complexes and the assertion will follow using Atiyah [5, Theorem 2.7.15] (a generalisation of Lemma 1) and the continuity of the K-theory (i.e. $\tilde{K}^*(\varprojlim\{X_\lambda\}) \cong \varinjlim\{\tilde{K}^*(X_\lambda)\}$ for an inverse system $\{X_\lambda\}$ of compact spaces).

Lemma 2 holds for compact X in the following version: as in Atiyah [5, Corollary 2.7.15] (or in [4]) one constructs a map $\alpha: SX \to A$ such that $\alpha^*: \tilde{K}^*(A) \to \tilde{K}^*(SX)$ is epic, where A is a – now possibly infinite – product of Grassmannians and suspensions of Grassmannians. Now A is the inverse limit of the finite products, thus $\tilde{K}^*(A)$ is a direct limit of free groups, and hence torsion free. [Indeed, $\tilde{K}^*(A)$ is even free, because bases can be arranged to be preserved by the bonding maps.]

For the construction of the Künneth sequence (K) for X, Y it is then enough to consider the torsion free geometric resolution

$$SX \xrightarrow{\alpha} A \xrightarrow{P\alpha} C\alpha \xrightarrow{Q\alpha} S^2X \xrightarrow{S\alpha} SA \to \dots \quad \text{of } X.$$

We have sketched the proof of

(3.3) **Theorem.** *There is a Künneth sequence*

$$\text{(K)} \quad 0 \to \tilde{K}^*(X) \otimes \tilde{K}^*(Y) \xrightarrow{\mu} \tilde{K}^*(X \wedge Y) \xrightarrow{\tau} \operatorname{Tor}(\tilde{K}^*(X), \tilde{K}^*(Y)) \to 0$$

which is exact and natural, for all X, Y. □

(3.4) Given a reduced multiplicative cohomology theory $\tilde{h}$, Araki and Toda [2, 3] and Maunder [14] studied the question, under which conditions the associated cohomology theory $\tilde{h}(\ ; n)$ with coefficients in $\mathbb{Z}_n$ is again multiplicative, i.e. whether there is an admissible multiplication

$$\mu_n: \tilde{h}^i(X; n) \otimes \tilde{h}^j(Y; n) \to \tilde{h}^{i+j}(X \wedge Y; n)$$

(see Araki and Toda [2, 3.3] for the definition).

In the case of K-theory the situation is favorable, again because $\tilde{K}$ is a "good" cohomology theory. We note

(3.5) **Proposition.** *For all $n \geqq 1$ there are admissible multiplications in $\tilde{K}(\ ;n)$.* □

For the proof of this result in Araki and Toda [2, Theorem 5.9] the restriction to finite CW complexes is again unnecessary. With μ we already have a map $\tilde{K}^i(X;n) \otimes \tilde{K}^j(Y;n) \to \tilde{K}^{i+j+4}(X \wedge Cn \wedge Y \wedge Cn)$ and the problem is to come back to $\tilde{K}^{i+j+2}(X \wedge Y \wedge Cn) = \tilde{K}^{i+j}(X \wedge Y; n)$ (compare Maunder [14, Theorem 4.1.1]). This is done by first constructing a map $g'_n: SCn \to S^2$ and then a natural left inverse $G_n = G_n^{X \wedge Y}$ for

$$(1 \wedge Qg'_n)^*: \tilde{K}^{i+j+4}(X \wedge Y \wedge S^2 Cn) \to \tilde{K}^{i+j+4}(X \wedge Y \wedge Cg'_n);$$

there is thirdly a stable homotopy class represented by some $g''_n: S^k Cg'_n \to S^k(Cn \wedge Cn)$. Then μ_n is (up to suspension isomorphisms and twisting the factors in the smash product) the composition of G_n, $(1 \wedge g''_n)^*$ and μ; all the properties of μ_n depend only on the properties of μ, G_n and g''_n, but do not depend on the spaces X, Y (Araki and Toda [2, 5.3]).

(3.6) *Remark.* By Araki and Toda [3, Theorem 6.1] there are exactly n admissible multiplications in $\tilde{K}(\ ;n)$, which are all associative by [3, Corollary 10.8]. We choose an arbitrary one among them and denote it from now on by μ_n.

Because of Lemma 3a) in Puppe [19] all we construct with this μ_n will be independent of the choice.

(3.7) An admissible multiplication in $\tilde{K}(\ ;n)$ is comparable with the reduction homomorphism (Araki and Toda [2, 3.3(Λ'_1)]). Comparability with the coefficient homomorphisms is not part of the axioms and it does not follow from the axioms in [2, 3.3] (unless one makes some additional assumptions, compare Araki and Toda [3, Proposition 11.8]).

But we have the following Lemma of Puppe, stating that the Bockstein homomorphism corrects this error in special cases.

(3.8) **Lemma.** $\beta_n \mu_n(\kappa_{n,kn} \otimes \kappa_{n,kn}) = \beta_n \kappa_{n,kn} \mu_{kn}$. □

The proof of Puppe [19, Lemma 3b)] holds also for compact spaces with an analogous remark as in (3.5) (only the co-Moore spaces have to be considered). Again one uses the important fact that $\eta^* = 0$ for the Hopf map η.

(3.9) If $x_n \in \tilde{K}^*(X;n)$ and $y_n \in \tilde{K}^*(Y;n)$, then $n\beta_n(x_n) = 0$ and $n\beta_n(y_n) = 0$, that is $[\beta_n(x_n), n, \beta_n(y_n)]$ is an element of $\mathrm{Tor}(\tilde{K}^*(X), \tilde{K}^*(Y))$. (We use here the definition of Tor given by MacLane [13, Chap. V, 6.], see also Fuchs [9, 62].) This defines a mapping $\tilde{K}^*(X;n) \times \tilde{K}^*(Y;n) \to \mathrm{Tor}(\tilde{K}^*(X), \tilde{K}^*(Y))$ which is obviously bilinear, hence it induces a unique map $\gamma_n: \tilde{K}^*(X;n) \otimes \tilde{K}^*(Y;n) \to \mathrm{Tor}(\tilde{K}^*(X), \tilde{K}^*(Y))$.

(3.10) **Lemma.** $\gamma_n = \tau \beta_n \mu_n$. □

For the proof we refer again to Puppe [19, Lemma 2] with the remark that only the properties of an admissible multiplication and the existence of torsion free geometric resolutions are used there. Hence the proof is also valid in our more general case of compact spaces. (Actually, there must be a minus sign in Lemma 1 and hence in Lemma 2 of [19]; but this does not matter to our purposes, because the sign can be incorporated in the definition of τ.)

4. Balanced Exact Sequences

(4.1) Let G be an (abelian) group and p a prime. For every ordinal σ we define the *Ulm-Kaplansky subgroups* $p^\sigma G$ of G by

$$p^\sigma G = G, \quad \text{for} \quad \sigma = 0,$$
$$p^\sigma G = p(p^{\sigma'} G), \quad \text{for} \quad \sigma = \sigma' + 1,$$
$$p^\sigma G = \bigcap_{\sigma' < \sigma} p^{\sigma'} G, \quad \text{for} \quad \sigma = \lim_{\sigma' < \sigma} \sigma'.$$

We get a decreasing series of subgroups

$$G = p^0 G \supseteqq pG \supseteqq \ldots \supseteqq p^\sigma G \supseteqq \ldots \supseteqq p^\lambda G,$$

where λ is the least ordinal such that $p^\lambda G = p^{\lambda+1} G$, and hence $p^\lambda G = \bigcap_{\sigma \leqq \lambda} p^\sigma G = d_p G$, the maximal p-divisible subgroup of G (Fuchs [9, 37]).

(4.2) Now we can define the *(generalised) p-heights:*

$$h_p^G(x) = \sigma, \quad \text{if} \quad x \in p^\sigma G \setminus p^{\sigma+1} G,$$
$$h_p^G(x) = \lambda, \quad \text{if} \quad x \in p^\lambda G.$$

[We write $h_p(x)$ instead of $h_p^G(x)$ when there is no danger of confusion.]

For these heights the following equalities and inequalities hold (Fuchs [9, 37]):

i) $h_p(px) \geqq h_p(x) + 1$,

ii) $h_p(x+y) \geqq \min(h_p(x), h_p(y))$,

iii) $h_p(x+y) = \min(h_p(x), h_p(y))$, if $h_p(x) \neq h_p(y)$,

iv) $h_p^H(\varphi(x)) \geqq h_p^G(x)$, for $\varphi: G \to H$.

(4.3) Let G be a group and $p, n \geqq 1$ relatively prime. If $nx = 0$ holds for $x \in G$, then x is divisible by p, i.e. $x \in pG$. Repeating the argument for $G' = pG$ it follows that $x \in d_p G$, hence $x \in p^\sigma G$ for any σ. And therefore $p^\sigma G = G$ for all σ if G is a q-group and $q \neq p$. Since $\operatorname{Tor}(A, B)$ is the direct sum of the groups $\operatorname{Tor}(t_q A, t_q B)$ where $t_q A$ and $t_q B$ are the q-primary components (of the torsion subgroups) of A and B respectively, the following results of Nunke [16, Theorem 1.5, Lemma 1.1] (see also Fuchs [9, Lemma 64.2, Lemma 64.3]) hold for arbitrary groups A, B and p prime, $n \geqq 1$.

(4.4) **Lemma.** i) $p^\sigma \operatorname{Tor}(A, B) = \operatorname{Tor}(p^\sigma A, p^\sigma B)$.

ii) $\operatorname{Tor}(A, B)[n] = \operatorname{Tor}(A[n], B[n])$. □

(4.5) For $A', A'' \subseteqq A$ and $B', B'' \subseteqq B$ we have $\operatorname{Tor}(A', B') \cap \operatorname{Tor}(A'', B'') = \operatorname{Tor}(A' \cap A'', B' \cap B'')$ by Nunke [17, Lemma 7] (see also Fuchs [9, 64, Exercise 4]). Therefore also the functor $p^\sigma G[n]=p^\sigma G\cap G[n]$ commutes with Tor and we note $p^\sigma \mathrm{Tor}(A,B)[n]=\mathrm{Tor}(p^\sigma A[n],p^\sigma B[n])$.

(4.6) Let p be again prime. We are now interested in the exactness properties of the functors p^σ. Let

(4.7) $0\to A\to B\xrightarrow{\pi} C\to 0$

be an exact sequence of groups. The sequence

(4.8) $0\to p^\sigma A\to p^\sigma B\xrightarrow{\pi|} p^\sigma C\to 0$

is in general no longer exact. It is always exact at $p^\sigma A$ for any σ, and it is always exact at $p^\sigma C$ for finite σ. Even for finite σ (4.8) need not be exact at $p^\sigma B$ in all cases; (4.8) is for all finite σ exact at $p^\sigma B$ (and hence everywhere) if and only if (4.7) is p-pure exact (Fuchs [9, Theorem 29.1a)]). The following notion is therefore a transfinite form of purity (Fuchs [10, 79, 80]).

a) The exact sequence (4.7) is called *p-balanced exact*, if (4.8) is exact for all σ.

b) The exact sequence (4.7) is called *balanced exact*, if (4.8) is exact for all σ and all prime p.

Later we shall use the following criterion for balancedness which is a modification of Lemma 80.2 in Fuchs [10].

(4.9) **Lemma.** *Let A, B be groups, C a torsion group, p prime and the sequence*

$$0\to A\to B\xrightarrow{\pi} C\to 0$$

pure exact. Then the sequence is p-balanced exact if and only if

$$\pi(p^\sigma B[p])=p^\sigma C[p]$$

holds for all σ.

Proof. We begin with two remarks.

a) If C is a p-group, we can drop the purity of the sequence and imitate the proof in Fuchs [10, Lemma 80.2].

b) We use the purity for the q-primary components C_q of C for $q\neq p$; more precisely we need only the q-purity for all $q\neq p$. But the p-purity follows any way from both characterisations and therefore it yields no loss of generality in applying the lemma.

The proof of Lemma 80.2 in Fuchs [10] applies also in our case, except to show the surjectivity of $\pi|: p^\sigma B\to p^\sigma C=p^\sigma(\oplus C_q)$ on the components C_q for $q\neq p$.

But if $c\in p^\sigma C$ and $q^k c=0$ for some k, there is a $b\in B$ with $\pi(b)=c$ and $q^k b=0$, because of the purity. And in fact, $b\in p^\sigma B$, according to the remarks in (4.3). Hence $\pi|$ is also epic on the q-primary components for $q\neq p$. □

Now we come to the next lemma, which can be found as Lemma 80.3 in Fuchs [10]. It is the crucial step in order to construct a splitting for a balanced exact sequence.

(4.10) **Lemma.** *In the diagram*

$$\begin{array}{ccccccc} & & N \subseteq \langle N, g\rangle \subseteq G & & \\ & & \psi \downarrow \quad \swarrow \psi' \qquad \downarrow \varphi & & \\ 0 \rightarrow A \rightarrow & B & \xrightarrow[\pi]{\qquad} & C \rightarrow 0 \end{array}$$

let the lower row be p-balanced exact and suppose

i) $\pi\psi = \varphi | N$,
ii) $h_p^B(\psi(x)) \geqq h_p^G(x)$, *for all* $x \in N$,
iii) $g \in G$ *and* $pg \in N$,
iv) $h_p^G(g) \geqq h_p^G(g+x)$, *for all* $x \in N$.

Then there is an extension ψ' *of* ψ *such that*

i') $\pi\psi' = \varphi | \langle N, g\rangle$,
ii') $h_p^B(\psi'(x')) \geqq h_p^G(x')$, *for all* $x' \in \langle N, g\rangle$. □

(4.11) The class of p-groups exhaustable by subgroups N_σ such that

$N_0 = 0$,

$N_{\sigma+1} = \langle N_\sigma, g_\sigma\rangle$, with N_σ, g_σ as in (4.10 iii) and iv)),

$N_\sigma = \bigcup_{\sigma' < \sigma} N_{\sigma'}$, for σ a limit ordinal

is exactly the class of p-groups with "nice composition series" (see Fuchs [10, 81] for the definition). It is important for us that all countable p-groups belong to this class.

(4.12) **Lemma.** *Let G be a countable p-group. Then G has a generating system* $\{e_k\}$, $k \in \mathbb{N}$, *with the properties*

i) $pe_0 = 0$,
ii) $pe_{k+1} \in \langle e_0, \ldots, e_k\rangle$,
iii) $h_p(e_{k+1}) \geqq h_p(e_{k+1} + x)$, *for all* $x \in \langle e_0, \ldots, e_k\rangle$.

Proof. (Compare Fuchs [10, Lemma 81.1].)

We take a countable set of generators $\{e_k\}$ and fill it up with all the multiples $p^n e_k, p^{n-1} e_k, \ldots, pe_k, e_k$ different from 0. A new enumeration implies obviously i) and ii). To obtain iii) we note that there is an $x_{k+1} \in \langle e_0, \ldots, e_k\rangle$ such that $h_p(e_{k+1} + x_{k+1})$ is maximal in the coset $e_{k+1} + \langle e_0, \ldots, e_k\rangle$, since $\langle e_0, \ldots, e_k\rangle$ is finite. Substituting $e_k + x_k$ for e_k we have a system of the desired type. □

(4.13) Given such a generating system $\{e_k\}$, we put $N_0 = 0$, $N_{k+1} = \langle e_0, \ldots, e_k\rangle$ and $g_k = e_k$; then a successive application of Lemma (4.10) gives

(4.14) **Theorem.** *A countable p-group is p-balanced projective, i.e. it has the projective property with respect to all p-balanced exact sequences.* □

(Compare Fuchs [10, 80, Exercise 11, 12].)

(4.15) **Corollary.** *A countable torsion group is balanced projective, i.e. it has the projective property with respect to all balanced exact sequences.* □

5. Splitting the Künneth Sequence

(5.1) After these preparations we are now in position to apply the theory of balanced exact sequences to the Künneth sequence.

Let X, Y be compact spaces, p a prime and s_{p^n}, $n \geqq 1$, the sections constructed in Theorem (2.8) for X respectively Y.

(5.2) **Lemma.** *If* $x \in \tilde{K}^{i+1}(X)[p^{n+1}]$, $y \in \tilde{K}^{j+1}(Y)[p^{n+1}]$, *then*

$$\beta_{p^n}\mu_{p^n}(s_{p^n}(px) \otimes s_{p^n}(py)) = p\beta_{p^{n+1}}\mu_{p^{n+1}}(s_{p^{n+1}}(x) \otimes s_{p^{n+1}}(y)).$$

Proof. The assertion is proved by the following computation using (2.1), Theorem (2.8ii)), Lemma (3.8), and Proposition (1.9ii)) one after the other

$$\begin{aligned}\beta_{p^n}\mu_{p^n}(s_{p^n}(px) \otimes s_{p^n}(py)) &= \beta_{p^n}\mu_{p^n}(s_{p^n}\kappa''_{p^n,p^{n+1}}(x) \otimes s_{p^n}\kappa''_{p^n,p^{n+1}}(y)) \\ &= \beta_{p^n}\mu_{p^n}(\kappa_{p^n,p^{n+1}}s_{p^{n+1}}(x) \otimes \kappa_{p^n,p^{n+1}}s_{p^{n+1}}(y)) \\ &= \beta_{p^n}\kappa_{p^n,p^{n+1}}\mu_{p^{n+1}}(s_{p^{n+1}}(x) \otimes s_{p^{n+1}}(y)) \\ &= p\beta_{p^{n+1}}\mu_{p^{n+1}}(s_{p^{n+1}}(x) \otimes s_{p^{n+1}}(y)). \quad \square\end{aligned}$$

(5.3) **Lemma.** *If* $x \in p^\sigma\tilde{K}^{i+1}(X)[p^n]$, $y \in p^\sigma\tilde{K}^{j+1}(Y)[p^n]$, *then*

$$\beta_{p^n}\mu_{p^n}(s_{p^n}(x) \otimes s_{p^n}(y)) \in p^\sigma\tilde{K}^{i+j+1}(X \wedge Y)[p^n].$$

Proof. We prove the lemma by transfinite induction on σ. The case $\sigma = 0$ is obvious and the case σ a limit ordinal is merely routine.

Assume $\sigma = \sigma' + 1$, and $x = px'$, $y = py'$ for some $x' \in p^{\sigma'}\tilde{K}^{i+1}(X)[p^{n+1}]$, $y' \in p^{\sigma'}\tilde{K}^{j+1}(Y)[p^{n+1}]$. By induction hypothesis we have

$$\beta_{p^{n+1}}\mu_{p^{n+1}}(s_{p^{n+1}}(x') \otimes s_{p^{n+1}}(y')) \in p^{\sigma'}\tilde{K}^{i+j+1}(X \wedge Y)[p^{n+1}],$$

hence applying Lemma (5.2) yields

$$\beta_{p^n}\mu_{p^n}(s_{p^n}(x) \otimes s_{p^n}(y)) = p\beta_{p^{n+1}}\mu_{p^{n+1}}(s_{p^{n+1}}(x') \otimes s_{p^{n+1}}(y')).$$

This completes the proof. □

The next lemma will enable us to apply Lemma (4.9), the criterion for balancedness, to the Künneth sequence.

(5.4) **Lemma.** *For the Künneth sequence it holds that*

$$\tau(p^\sigma\tilde{K}^\#(X \wedge Y)[p^n]) = p^\sigma \mathrm{Tor}(\tilde{K}^\#(X), \tilde{K}^\#(Y))[p^n].$$

Proof. Since a homomorphism neither decreases heights (4.2iv)) nor increases orders the inclusion $\subseteqq$ is clear. By (4.5) we have

$$T = p^\sigma \mathrm{Tor}(\tilde{K}^\#(X), \tilde{K}^\#(Y))[p^n] = \mathrm{Tor}(p^\sigma\tilde{K}^\#(X)[p^n], p^\sigma\tilde{K}^\#(Y)[p^n]),$$

hence T is generated by elements of the form $[x, p^m, y]$ with $x \in p^\sigma\tilde{K}^\#(X)[p^n]$, $y \in p^\sigma\tilde{K}^\#(Y)[p^n]$, x and y homogeneous in $\tilde{K}^\#(X)$ and $\tilde{K}^\#(Y)$ respectively and

$m \leqq n$. To prove the opposite inclusion $\supseteqq$ it is therefore enough to find preimages in $p^\sigma \tilde{K}^\#(X \wedge Y)[p^n]$ for such generators of T. But by Lemma (5.3)

$$\beta_{p^m}\mu_{p^m}(s_{p^m}(x)\otimes s_{p^m}(y)) \in p^\sigma \tilde{K}^\#(X \wedge Y)[p^m] \subseteqq p^\sigma \tilde{K}^\#(X \wedge Y)[p^n]$$

and by Lemma (3.10) we get also

$$\begin{aligned}\tau\beta_{p^m}\mu_{p^m}(s_{p^m}(x)\otimes s_{p^m}(y)) &= \gamma_{p^m}(s_{p^m}(x)\otimes s_{p^m}(y))\\ &= [\beta_{p^m}s_{p^m}(x), p^m, \beta_{p^m}s_{p^m}(y)]\\ &= [\tilde{\beta}_{p^m}s_{p^m}(x), p^m, \tilde{\beta}_{p^m}s_{p^m}(y)]\\ &= [x, p^m, y]. \quad \square\end{aligned}$$

Now we deduce our main results.

If we put $\sigma = 0$ in Lemma (5.4) we get by Fuchs [9, Theorem 29.1]

(5.5) **Theorem.** *The Künneth sequence is pure exact for all compact X, Y.* □

This theorem can already be found implicitly in the papers mentioned in the introduction: Mislin [15, Theorem 5.4, Lemma 1.4], Puppe [19, Satz 1], and Anderson [1, Corollary 3.1]. As an immediate consequence one gets the splitting if one of the spaces is a finite CW complex; compare (5.9).

Using this last theorem and Lemma (5.4) for $n = 1$ we derive from Lemma (4.9) a theorem which contains it as a special case.

(5.6) **Theorem.** *The Künneth sequence is balanced exact for all compact X, Y.* □

(5.7) If Z is a compact metric space, it is the inverse limit of an inverse sequence of finite CW complexes. Because of the continuity of the K-theory $\tilde{K}^\#(Z)$ is the direct limit of a direct sequence of finitely generated groups, hence countable.

If X, Y are compact metric spaces, $\tilde{K}^\#(X)$, $\tilde{K}^\#(Y)$ are both countable groups and therefore $\mathrm{Tor}(\tilde{K}^\#(X), \tilde{K}^\#(Y))$ is countable. Applying Corollary (4.15) completes the proof of

(5.8) **Theorem.** *The Künneth sequence splits (unnaturally) for all compact metric X, Y.* □

(5.9) *Concluding Remark.* We come back to our remark in (5.5). To derive from Theorem (5.5) that the Künneth sequence splits if one of the spaces is a finite CW complex the assumption is – strictly speaking – not X (or Y) to be a finite CW complex, but $\mathrm{Tor}(\tilde{K}^\#(X), \tilde{K}^\#(Y))$ to be a direct sum of cyclic groups (= pure projective, Fuchs [9, Theorem 30.2]). This, for instance, holds if the torsion subgroup of $\tilde{K}^\#(X)$ is a direct sum of cyclic groups and therefore it is enough that X is a finite CW complex.

Correspondingly, to derive Theorem (5.8) from Theorem (5.7) the assumption is $\mathrm{Tor}(\tilde{K}^\#(X), \tilde{K}^\#(Y))$ to be totally projective (= balanced projective, Fuchs [10, Theorem 82.3]). (See Fuchs [10, 81–83] for the definition of totally projective groups and equivalent notions.) Since countable torsion groups are totally projective [Corollary (4.15)], we assume X, Y to be compact metric.

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