STABLE SPLITTINGS OF MAPPING SPACES

C.-F. Bödigheimer

0. Introduction.

In this note we elaborate on two observations concerning configuration spaces; they will lead to a stable splitting of certain mapping spaces into infinite bouquets of simpler spaces.

Let K be a finite complex, K_o a subcomplex, and X a connected CW-complex. Then choose a smooth, compact and parallelizable m-manifold M with a submanifold M_o such that the pairs (K,K_o) and (M,M_o) are homotopy equivalent. For the space $map(K,K_o;S^mX)$ of based maps from K/K_o to S^mX we prove

PROPOSITION 1.

There is a stable equivalence

$$map(K,K_o;S^mX) \underset{s}{\simeq} \bigvee_{k=1}^{\infty} \mathcal{D}_k \quad ;$$

the spaces $\mathcal{D}_k$ *depend on* M, M_o *and* X, *in particular* $\mathcal{D}_1 = (M\setminus M_o, \partial M\setminus M_o)\wedge X$.

Several special cases of this proposition are well-known.

EXAMPLE 1. $K = M = [0,1]$, $K_o = M_o = \{0,1\}$.
The proposition gives a splitting of the suspension spectrum $S^\infty\Omega SX$; a refinement of the proof would yield the splitting of $S\Omega SX$ found by Milnor [17], see Remark 3.

EXAMPLE 2. $K = M = \mathbb{D}^m$, $K_o = M_o = \partial\mathbb{D}^m$.
This is the stable splitting of $\Omega^m S^m X$ found by Snaith [20].

EXAMPLE 3. $K = M = S^1$, $K_o = M_o = \emptyset$.
A stable splitting of the free loop space ΛSX of SX has recently been obtained by Goodwillie (unpublished).

EXAMPLE 4. $K = M = S^{m-1} \times [0,1]$, $K_o = M_o = S^{m-1} \times \{0,1\}$.
This example gives a stable splitting of $\bar{\Omega}^m S^m X$, the space of maps $f : S^m \longrightarrow S^m X$ such that $f(s_o) = f(-s_o) = *$, where s_o and $*$ are the basepoints; it is particularly interesting for $\mathbb{Z}/2$- and S^1-equivariant homotopy theory, (N.B. $\bar{\Omega}^m S^m X \simeq \Omega^m S^m X \times \Omega S^m X$.)

EXAMPLE 5. $K = \mathbb{D}^m$, $K_o = \partial\mathbb{D}^m$, $M = \mathbb{D}^m \times [0,1]$, $M_o = \partial\mathbb{D}^m \times [0,1]$.
In this case we obtain - also for non-connected X - a stable splitting of $\Omega^m S^{m+1} X$; it is different but equivalent to the corresponding one replacing X by SX in Example 2.

EXAMPLE 6. $K = M = G$ a compact Lie group of dimension m, $K_o = M_o = \emptyset$.
Here the mapping space is the space of all unbased maps from G to $S^m X$.

EXAMPLE 7. K = point, $K_o = \emptyset$, $M = \mathbb{D}^m$, $M_o = \emptyset$.
We have $\mathrm{map}(K,K_o;S^m X) = S^m X = \mathcal{D}_1$, all other $\mathcal{D}_k$ are contractible.

EXAMPLE 8. In general one can choose an embedding $K \subset \mathbb{R}^m$ of K, a regular neighbourhood M, a submanifold M_o with $K_o \subset M_o$ and a deformation retraction of pairs $r_t : (M,M_o) \longrightarrow (K,K_o)$. Hence $\mathrm{map}(K,K_o;S^m X)$ always stably splits into a bouquet, if m is at least the embedding dimension of K.

Such splittings are usually obtaiend by splitting appropriate configuration space models for the mapping spaces. In Section 1 we will define these models. In Section 2 we observe that (under certain connectivity assumptions) they are equivalent to mapping spaces. In Section 3 we observe that these models split stably, and we conclude Proposition 1. In Section 4 we list some properties of the splittings.

We do not claim any originality. In fact, all the constructions and proofs either can be found in the literature (e.g. [2], [5], [16] and [20]) are well-known to the experts. Only the importance of such splittings may justify the publication of a unified approach.

The author is indepted to the referee and to F. Cohen for demanding more details to make the following pages more self-contained and readable.

1. The Configuration Spaces.

Let N be a smooth m-manifold, N_o a submanifold (closed as a subspace), and X a CW-complex with basepoint $*$. We denote by $C(N,N_o;X)$ the space of finite configurations of particles in N with parameters (or labels) in X, which are annihilated in N_o or for vanishing; more precisely, let $\tilde{C}(N,k) = \{(z_1,\ldots,z_k) \in N^k \mid z_i \neq z_j \text{ for } i \neq j\}$ be the space of ordered (unlabeled) configurations of k points in N; then $C(N,N_o;X)$ is the quotient of $\coprod_{k=1}^{\infty} \tilde{C}(N,k) \times X^k$ by the following identifications:

(1.1) actions of the symmetric groups Υ_k

$$(z_1,\ldots,z_k;x_1,\ldots,x_k) \sim (z_{s(1)},\ldots,z_{s(k)};x_{s(1)},\ldots,x_{s(k)}) \text{ for } s \in \Upsilon_k;$$

(1.2) annihilation of particles with parameter $*$

$$(z_1,\ldots,z_k;x_1,\ldots,x_k) \sim (z_1,\ldots,z_{k-1};x_1,\ldots,x_{k-1}) \text{ if } x_k = *;$$

(1.3) annihilation of particles in N_o

$$(z_1,\ldots,z_k;x_1,\ldots,x_k) \sim (z_1,\ldots,z_{k-1};x_1,\ldots,x_{k-1}) \text{ if } z_k \in N_o.$$

Because of (a) we will write a configuration $\xi \in C = C(N,N_o;X)$ as a formal sum $\xi = \Sigma z_i x_i$ bearing in mind that C is a subspace of the infinite symmetric product $SP_\infty((N/N_o) \wedge X)$; then (1.2) and (1.3) can be replaced by: $zx = 0$ if $x = *$ or $z \in N_o$, respectively, where 0 denotes the basepoint in C (which is represented by any $\xi = \Sigma z_i x_i$ such that for all i, $x_i = *$ or $z_i \in N_o$ holds).

Such configuration spaces have been extensively studied by Fadell-Neuwirth [8] for $N_o = \emptyset$ and $X = S^o$, by Mc Duff [16] for $N_o = \partial N$ and $X = S^o$, and by Cohen-Taylor [2] for $N_o = \emptyset$.

<u>EXAMPLE 9.</u> $C(\mathbb{R}^m;X) = C(\mathbb{R}^m,\emptyset;X)$ are the well known configuration spaces of May [13] and Segal [19]. $C(\mathbb{R}^1;X)$ is homotopy equivalent to the James construction [9].

The length k of a configuration $\xi = \sum_{i=1}^{k} z_i x_i$ induces a natural filtration of C by closed subspaces $C_k(N,N_o;X) = (\coprod_{i=1}^{k} \tilde{C}(N,k) \times X^k)/\sim$. The inclusion $C_{k-1} \longrightarrow C_k$ is a cofibration, because $N_o \longrightarrow N$ and $* \longrightarrow X$ are. C_o consists of 0 only, and C_1 is $(N,N_o) \wedge X$.

If the pair (N,N_o) or X is connected then each particle z_i of a configuration ξ can be moved to N_o or its parameter x_i can be moved to $*$; therefore ξ can be moved to 0, i.e. C is connected. If N is connected, $N_o = \emptyset$ and $X = S^o$, then the strata $C_k - C_{k-1} = \tilde{C}(N,k)/\gamma_k = C(N,k)$ of $C = C(N) = C(N,\emptyset;S^o)$ are the connected components of C.

So far we have not used that N is a manifold - indeed N might have been any space; in particular, $C(\mathbb{R}^\infty;X)$ will be of importance to us (see Example 13 and Section 3).

<u>EXAMPLE 10.</u> The connected components of $C(\mathbb{R}^\infty)$ are the classifying spaces of the symmetric groups; those of $C(\mathbb{R}^2)$ are the classifying spaces of Artin's braid groups.

<u>EXAMPLE 11.</u> $C(\mathbb{D}^m, \partial\mathbb{D}^m; X)$ is homotopy equivalent to $S^m X$, see [16; p. 95].

The construction C is a homotopy functor in X, but only an isotopy functor in (N, N_o).

So, for example, the inclusion $N \setminus \partial N \longrightarrow N$ induces a homotopy equivalence $C(N \setminus \partial N, N_o \setminus \partial N; X) \longrightarrow C(N, N_o; X)$. The excision property $C(N, N_o; X) \cong C(N \setminus U, N_o \setminus U; X)$ for $U \subset N_o$ and U open in N, and the product property $C(N, N_o; X) \cong C(N', N' \cap N_o; X) \times C(N'', N'' \cap N_o; X)$ for $N = N' \cup N''$ and $N' \cap N'' \subset N_o$ follow easily from the definition. The crucial property of C is contained in the following lemma.

<u>Lemma.</u>

<u>Let</u> $H \subset N$ <u>be an</u> m-<u>dimensional submanifold</u>. <u>Then the isotopy cofibration</u>

$$(H, H \cap N_o) \longrightarrow (N, N_o) \xrightarrow{q} (N, H \cup N_o)$$

<u>induces a quasifibration</u>

$$C(H, H \cap N_o; X) \longrightarrow C(N, N_o; X) \xrightarrow{Q} C(N, H \cup N_o; X)$$

<u>provided</u> $(H, H \cap N_o)$ <u>or</u> X <u>is connected</u>.

<u>Proof:</u> Except for the presence of a parameter space X the proof is that of [16; Proposition 3.1]; we list the various steps.

(1) We filter the base space $B = C(N, H \cup N_o; X)$ by $B_k = C_k(N, H \cup N_o; X)$, and the total space $E = C(N, N_o; X)$ by $E_k = Q^{-1}(B_k)$, and we denote the fibre by $F = C(H, H \cap N_o; X)$.

(2) Observe that for each k there is homeomorphism

$$h_k : E_k \setminus E_{k-1} \cong (B_k \setminus B_{k-1}) \times F \quad \text{over} \quad B_k \setminus B_{k-1} .$$

(3) A tubular neighbourhood U of H defines for each k a neighbourhood U_k of B_k in B_{k+1}, and an isotopy retraction $r : U \rightarrow H$ induces

retractions $r_k : U_k \to B_k$, and retractions $\bar{r}_k : Q^{-1}(U_k) \to Q^{-1}(B_k) = E_k$ lying over r_k.

(4) For every $b \in U_k$ the induced map

$$F \xleftarrow[h_{k+1}|]{\cong} Q^{-1}(b) \xrightarrow[\bar{r}_k|]{} Q^{-1}(r_k(b)) \xrightarrow[h_k|]{\cong} F$$

is a homotopy equivalence (precisely because $(H, H \cap N_o)$ or X is connected).

It follows from the Dold-Thom criterion [8 ; 2.10, 2.15, 5.2] that Q is a quasifibration. □

2. The Section Spaces

The space $C(N,N_o;X)$ is under certain connectivity conditions equivalent to the space of sections of a certain bundle with fibre S^mX, and whence sometimes equivalent to a space of maps into S^mX. To make this precise let W be any smooth m-manifold without boundary which contains N (for example, $W = N$ if $\partial N = \emptyset$, or $W = N \cup (\partial N \times [0,1[)$ otherwise); if $\hat{T}(W)$ denotes the fibrewise compactification of the tangent bundle $T(W)$ of W, then define $\hat{T}(W;X) = \hat{T}(W) \wedge_\tau X$ to be fibrewise smash product of $\hat{T}(W)$ and X; this is a new bundle $\hat{\tau} : \hat{T}(W;X) \longrightarrow W$ with fibre S^mX.

The inclusion of the basepoint into each fibre yields a section g_∞ of $\hat{\tau}$. For $A_o \subset A \subset W$ let $\Gamma(A,A_o;X)$ denote the space of sections of $\hat{\tau}$ which are defined on A and agree with g_∞ on A_o; it is equipped with the (compactly generated topology induced by the) compact-open topology. (For example, if $X = S^o$ then $\hat{T}(W,S^o) = \hat{T}(W)$ and the sections are the vector fields with possible poles.)

The main theorem about configuration spaces on manifolds is the following duality.

PROPOSITION 2.

For compact N there is a map $\gamma : C(N,N_o;X) \longrightarrow \Gamma(W\setminus N_o, W\setminus N;X)$, which is a (weak) homotopy equivalence provided (N,N_o) or X is connected.

Proof: The proof is essentially contained in Mc Duff [16 ; Theorem 1.4] or [15]. For convenience we indicate the various steps.

(1) Following ideas of Gromov the map γ is defined as in [16 ; p. 95], or as in [15 ; p. 90] using Example 11, we have $\gamma(0) = g_\infty$.

(2) We start to prove the assertion with the case of (N,N_o) being a handle $(\mathbb{D}^m, \mathbb{D}^k \times S^{m-k-1})$ of index k. First, the assertion is true for k = 0 by Example 11. Consider for $k = 1,2,\dots,m$ in $I^k = [0,1]^m$ the subspace I^m_k of all $y = (y^i,\dots,y^m)$ such that $y^i = 0$ or $y^i = 1$ for some $i = k+1,\dots,m$, or $y^k = 1$; set $H^k = [0,1]^{k-1} \times [0,\frac{1}{2}] \times [0,1]^{m-k}$. In the sequence

(3) $(H_k, H_k \cap I^m_k) \longrightarrow (I^m, I^m_k) \longrightarrow (I^m, H_k \cup I^m_k)$ the left hand pair is a handle of index k, the right hand pair is a handle of index k-1. We apply $C(\ ;X)$ to (3) and obtain by the above lemma a quasifibration for $k = 1,\dots,m-1$ if X is arbitrary, and in addition for k = m if X is arbitrary, and in addition for k = m if X is connected. We apply $\Gamma(\ ;X)$ to the complements in $W = \mathbb{R}^m$ of (3) and obtain a fibration; γ maps the quasifibration to the fibration. Notice that both total spaces are contractible. Hence we conclude by induction the assertion for all handles of index $k = 0,1,\dots,m-1$ if X is arbitrary, and in addition for the handle of index m if X is connected.

(4) For the case $(N,\partial N)$ choose a handle decomposition of N, and if $(N,\partial N)$ is connected choose one without handles of index m. Attaching a new handle gives a quasifibration for C and a fibration for Γ, γ mapping

one to the other. Induction on the number of handles proves the assertion for $(N,\partial N)$.

(5) For the case (N,N_o) with $N_o \subset \partial N$ we choose a complementary submanifold $L \subset \partial N$, i.e. $L \cup N_o = \partial N$ and $L \cap N_o = \partial L = \partial N_o$. We attach a closed collar to N, $\bar{N} = N \cup (N \times [0,1])$, and consider the sequence

(6) $(\bar{L},\bar{L} \cap \bar{N}_o) \longrightarrow (\bar{N},\bar{N}_o) \longrightarrow (\bar{N},\bar{L} \cup \bar{N}_o)$ with $\bar{L} = L \times [0,1]$ and $\bar{N}_o = N_o \times [0,1]$. The assertion is true for the right hand pair by (4) since $(\bar{N},\bar{L} \cup \bar{N}_o) = (\bar{N},\partial\bar{N}) \cong (N,\partial N)$. As before, the assertion will follow for $(\bar{N},\bar{N}_o) \cong (N,N_o)$ if we can prove it for $(\bar{L},\bar{L} \cap \bar{N}_o) = (\bar{L},\partial\bar{L}) = (L,\partial L) \times [0,1]$.

(7) For this case we use the sequence

(8) $(L,\partial L) \times [0,1] \longrightarrow (L,\partial L) \times ([0,2],\{2\}) \longrightarrow (L \times [0,2],\partial(L \times [0,2]))$. The assertion is true for the right hand pair by (4); it is true for the middle pair, since this gives contractible spaces. Hence the assertion follows for the left hand pair.

(9) For the case of an arbitrary submanifold $N_o \subset N$ we replace N_o by closed tubular neighbourhood and then remove the interior of this neighbourhood. By isotopy invariance and excision property both manipulations leave the homotopy type of C unaltered. But now we are in case (5). □

<u>EXAMPLE 12</u>. (Example 8 continued). Under the assumptions of Proposition 1 set $N = M \setminus M_o$ and $N_o = \partial M \setminus M_o$, and $W = M \cup (\partial M \times [0,1[)$ if $\partial M \neq \emptyset$, or $W = M$ if $\partial M = \emptyset$. As a corollary we have

$$\begin{aligned} C(M \setminus M_o,\partial M \setminus M_o;X) &\simeq \Gamma(W \setminus (\partial M \setminus M_o),W \setminus (M \setminus M_o);X) \text{ by Proposition 12} \\ &= \Gamma(W \setminus \partial M) \cup M_o,(W \setminus M) \cup M_o;X) \\ &= \Gamma(M \setminus \partial M,M_o \setminus \partial M;X) \text{ by excision} \\ &\simeq \Gamma(M,M_o;X) \text{ by extension over } \partial M \\ &\cong \mathrm{map}(M,M_o;S^mX) \text{ by parallelizability} \\ &\simeq \mathrm{map}(K,K_o;S^mX) \end{aligned}$$,

where we should replace M_o by an open tubular neighbourhood to ensure compactness of $M \setminus M_o$.

EXAMPLE 13 (Example 2, 9 and 10 continued). If $N = \mathbb{D}^m$, $N_o = \emptyset$ and $W = \mathbb{R}^m$, then γ is the well-known approximation $C(\mathbb{R}^m;X) \simeq C(\mathbb{D}^m;X) \longrightarrow \mathrm{map}(\mathbb{R}^m, \mathbb{R}^m \setminus \mathbb{D}^m; S^m X) \simeq \Omega^m S^m X$ of May [13] and Segal [19]. Passing to the limit over m yields $\gamma^\infty : C(\mathbb{R}^\infty;X) \longrightarrow \Omega^\infty S^\infty X = \quad (X)$. See also Vogel [21].

Remark 1. For $C(M \setminus M_o, \partial M \setminus M_o;X)$ to be a model for $\mathrm{map}(K,K_o;S^m X)$ it is obviously enough that (M,M_o) is relatively compact and relatively parallelizable; but more important is that X need not be connected if $(M \setminus M_o, \partial M \setminus M_o)$ happens to be connected, see e.g. Example 5. In general, γ approximates the homology of the section space, see [16]; so in case $\partial N \neq \emptyset$, γ is a completion of homology modules over $H_*(\Omega\,\mathrm{map}(\partial N;S^m X))$. An interesting example is $C(\mathbb{R}P^m)$, since $\Gamma(\mathbb{R}P^m) = \Gamma(\mathbb{R}P^m;S^o)$ is the space of self-maps of S^m which are equivariant with respect to the antipodal action.

3. The Stable Splittings.

In [20] Snaith has obtained a stable splitting of $\Omega^m S^m X$ using the models $C(\mathbb{R}^m;X)$. Since then several authors have given very elegant proofs of this result, see F. Cohen [5], R. Cohen [6], Cohen-May-Taylor [3], May-Taylor [14], Vogt [22]. Our construction of a stable splitting of $C = C(N,N_o;X)$ is almost verbatim taken from [5].

Let $D_k = D_k(N,N_o;X)$ denote the filtration quotients C_k/C_{k-1} and consider the bouquet $V = V(N,N_o;X) = \bigvee_{k=1}^{\infty} D_k$ with the filtration given by $V_k = \bigvee_{j=1}^{k} D_j$.

Next we define the "power set map" $P : C \longrightarrow C(\mathbb{R}^\infty;V)$. Take some

$\xi = \sum_i z_i x_i \in C$ and a (non-empty) subset $\alpha = \{i_1,\ldots,i_k\}$ of the index set $I(\xi)$ of ξ. Define Z_α to be the (unlabeled) configuration $Z_\alpha = \sum_{j=1}^{k} z_{i_j}$ consisting of all z_i in ξ such that $i \in \alpha$; Z_α is in $\tilde{C}(N,k)/\Sigma_k = C(N,k)$ which is an km-manifold; we choose an embedding of their disjoint union $C(N) = \coprod_{k=1}^{\infty} C(N,k)$ into $\mathbb{R}^\infty$, and let $\bar{Z}_\alpha \in \mathbb{R}^\infty$ denote the image of Z_α under this embedding. Correspondingly, define ξ_α to be the subconfiguration $\xi_\alpha = \sum_{j=1}^{k} z_{i_j} x_{i_j}$ of ξ consisting of all labeled particles $z_i x_i$ of ξ such that $i \in \alpha$; ξ_α is in $C_k = C_k(N,N_o;X)$; using the quotient map $C_k \longrightarrow D_k$ and the inclusion $D_k \longrightarrow V$ we let $\bar{\xi}_\alpha \in V$ denote the image of ξ_α under the composition of these two maps. Finally, we define $P(\xi) = \sum_\alpha \bar{Z}_\alpha \bar{\xi}_\alpha$ in $C(\mathbb{R}^\infty;V)$ where the sum is over all subsets of $I(\xi)$.

Notice that the $\bar{Z}_\alpha$ are mutually different since two of the same length k have already different Z_α in $C(N,k)$, and the various $C(N,k)$ are disjointly embedded into $\mathbb{R}^\infty$. P is continous since it is well-defined: (1.1) is respected because a permutation of $I(\xi)$ only permutes the new indices α; (1.2) and (1.3) are respected because if $z_i \in N_o$ or $x_i = *$, then, for any α such that $i \in \alpha$, $\bar{\xi}_\alpha$ is the basepoint in D_k and in V, hence $\bar{Z}_\alpha \bar{\xi}_\alpha = 0$ in $C(\mathbb{R}^\infty;V)$.

Now let $\sigma : S^\infty C \longrightarrow S^\infty V$ denote the adjoint of the composition $\gamma^\infty \circ P : C \longrightarrow C(\mathbb{R}^\infty;V) \longrightarrow Q(V) = \Omega^\infty S^\infty V$ with γ^∞ as in Example 13.

<u>PROPOSITION 3.</u>
σ <u>is a stable equivalence</u> $C(N,N_o;X) \longrightarrow \bigvee_{k=1}^{\infty} D_k(N,N_o;X)$ <u>for any</u> (N,N_o) <u>and</u> X.

<u>Proof:</u> σ obviously preserves the filtration and we have a commutative lower square in the diagram

$$\begin{array}{ccc}
S^\infty(C_k/C_{k-1}) & = & S^\infty(V_k/V_{k-1}) \\
\uparrow & & \uparrow \\
S^\infty C_k & \xrightarrow{\sigma} & S^\infty V_k \\
\uparrow & & \uparrow \\
S^\infty C_{k-1} & \xrightarrow{\sigma} & S^\infty V_{k-1}
\end{array} ,$$

whereas the upper square is only homotopy commutative. Since the vertical sequences are cofibrations and since $C_1 = V_1$ the assertion follows by induction on k. □

Proof of Proposition 1 (Example 7 and 11 continued). The stable splitting of $\mathrm{map}(K,K_o;S^m X)$ now follows from that of $C(M \setminus M_o, \partial M \setminus M_o;X)$. The spaces $\mathcal{D}_k$ are $D_k(M \setminus M_o, \partial M \setminus M_o;X)$, in particular we have $\mathcal{D}_1 = C_1(M \setminus M_o, \partial M \setminus M_o;X) = (M \setminus M_o, \partial M \setminus M_o) \wedge X$. □

EXAMPLE 14 (Example 2, 8 and 12 continued). The splitting we obtain for $K = M = \mathbb{D}^m$ and $K_o = M_o = \partial\mathbb{D}^m$ is the Snaith splitting of [20].

Remark 2. In the proof of Proposition 3 we did not use that N is a manifold; the proof covers also the case of $C(\mathbb{R}^\infty;X)$ which is equivalent to $\Omega^\infty S^\infty X$ if X is connected. A stable splitting of $\Omega^\infty S^\infty X$ was first obtained by Kahn, see [1], [10], [11] and [12]. Furthermore, we did not use that (N,N_o) or X is connected. This and Remark 1 shows that Proposition 1 is more generally true than stated, see e.g. Example 5.

Remark 3. A splitting of $S\Omega SX$ is achieved by refining the power set map to a map $P : C = C(\mathbb{R};X) \longrightarrow C(\mathbb{R};V(\mathbb{R};X))$; the order of the particles z_i on the real line induces a lexicographic order of the sets α, and the hereby induced order of the Z_α is used to define particles $\bar{Z}_\alpha$ in $\mathbb{R}$ instead of $\mathbb{R}^\infty$.

4. Naturality and Homology.

Assume we have two situations as in the introduction, a map $f : (K,K_o) \longrightarrow (K',K'_o)$ together with an embedding $F : (M,M_o) \longrightarrow (M',M'_o)$ making the obvious diagram commutative, $m = m'$ and $X = X'$. Then f induces $f^* : \mathrm{map}(K',K'_o;S^mX) \longrightarrow \mathrm{map}(K,K_o;S^mX)$, while F induces $F^* : C(M' \setminus M'_o, \partial M \setminus M'_o;X) \longrightarrow C(M \setminus M_o, \partial M \setminus M_o;X)$ and $F^*_k : D_k(M' \setminus M'_o, \partial M' \setminus M'_o;X) \longrightarrow D_k(M \setminus M_o, \partial M \setminus M_o;X)$. The approximation map γ of Proposition 2 and the splitting map σ of Proposition 3 commute with these induced maps.

Examples for such maps f are the inclusions $K_o \longrightarrow K$ and $K \longrightarrow (K,K_o)$, the inclusion of a bottom cell of K and the pinch map onto a top cell of K.

γ and σ are natural with respect to the suspension $\mathrm{map}(K,K_o;S^mX) \longrightarrow \mathrm{map}(S(K,K_o);S^{m+1}X)$, which for C and V is induced by the equatorial inclusion $(M,M_o) \longrightarrow (M,M_o) \times ([0,1],\{0,1\})$.

An analysis of the splitting map σ reveals that each of the spaces $\mathcal{D}_k$ is already after a finite number of suspensions a retract of C. An upper bound for the smallest number is given by the embedding dimension of $C(N,k)$. In our standard situation of Example 7 we have $N = M \setminus M_o$ as a submanifold of $\mathbb{R}^m$, so $\mathcal{D}_1 = (M \setminus M_o, \partial M \setminus M_o) \wedge X$ is a retract of $\mathrm{map}(K,K_o;S^mX)$ after at most m suspensions.

The (stable) projection onto this first summand $S^m\mathrm{map}(K,K_o;S^mX) \simeq S^mC \longrightarrow S^m\mathcal{D}_1 = S^m(M \setminus M_o, \partial M \setminus M_o) \wedge X$ induces the homology slant product $H_q(\mathrm{map}(K,K_o;S^mX)) \longrightarrow \bigoplus_j H^{j-q}(K,K_o;H_{j-m}(X))$.

For $X = S^o$ this homomorphism has been proved by Moore [18] to be an isomorphism if $q < 2(m - H\dim(K,K_o))$ which is twice the connectivity of the mapping space.

Studying the spaces $\mathcal{D}_k$ (which are Thom spaces for X a sphere) is a possible approach to the homology of the mapping spaces; we will return to this in a further article.

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Sonderforschungsbereich 170
"Geometrie und Analysis"
Bunsenstraße 3/5

3400 Göttingen, BRD

Added in proof:

R. Cohen has independently found a model and a stable splitting for ΛSX (see his "A Model for the Free Loop Space of a Suspension", to appear).