Hurwitz continued fractions and Ruelle's transfer operator

Tobias Mühlenbruch
Joint work with D. Mayer and F. Strömberg

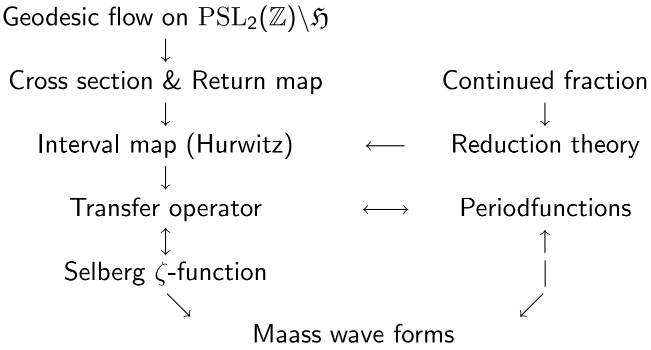
Institute of Theoretical Physics
TU Clausthal

tobias.muehlenbruch@tu-clausthal.de
http://home.tu-clausthal.de/∼tmu/

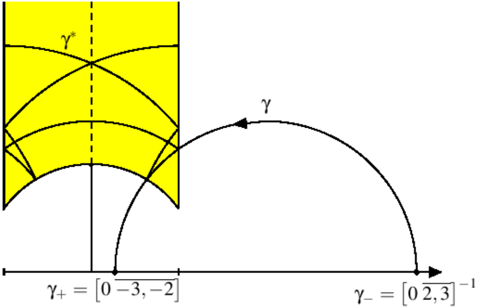
Minisymposium on
Automorphic forms and their applications
Bonn, 18–20 September 2006

TU Clausthal

A simplified overview



Let $\mathrm{PSL}_2(\mathbb{Z}) = \langle S, T \rangle$
with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$,
with $(ST)^3 = 1$.



We denote geodesics γ on $\mathfrak{H}$ (resp. $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$) by its base points:

$$\gamma = (\gamma_-, \gamma_+).$$

Example

In the example we see the geodesic
$\gamma = ([0; \overline{-3, -2}], [0; \overline{-2, -3}]^{-1}) \approx (0.422\ldots, 1.57\ldots)$.

Definition (Hurwitz continued fractions)

We identify a sequence of integers, $a_0 \in \mathbb{Z}$, and $a_1, a_2, \ldots \in \mathbb{Z}^\star$ with

$$x = T_0^a\, ST^{a_1}\, ST^{a_2} \cdots 0 = a_0 + \cfrac{-1}{a_1 + \cfrac{-1}{a_2 + \frac{-1}{\cdots}}}$$

and say that it is a

- *non-regular (formal) CF*, $[a0; a1, a2, \ldots]$ in general.
- *regular CF*, $[a0; a1, a2, \ldots]$, if it does not contain "forbidden blocks": no ± 1 appear and if $a_i = \pm 2$ then $a_{i+1} \lesssim 0$.

$\pi = [3; -7, 16, 294, 3, 4, 5, 15, \ldots]$ and $e = [3; 4, 2, -5, -2, 7, 2, -9, \ldots]$

Equivalent points

x and y are equivalet $:\Leftrightarrow$ there exist a $g \in \mathrm{PSL}_2(\mathbb{Z})$ such that $gx = y \Leftrightarrow$

- the CF of x and y have the same tail or
- the CF of x and y have tail $[\,\overline{3}\,]$ and $[\,\overline{-3}\,]$.

The generating map f

- Let $(x) = \left\lfloor x + \frac{1}{2} \right\rfloor$ be the nearest integer x and put $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$.
- The *generating map* for the CF of x is

$$f: I \to I; \quad x \mapsto \frac{-1}{x} - \left(\frac{-1}{x}\right) = \frac{-1}{x} - \left\lfloor \frac{-1}{x} + \frac{1}{2} \right\rfloor.$$

- If we set $y_1 = -\frac{1}{x}$ then the CF $x = [a_0; a_1, \ldots]$ are computed by

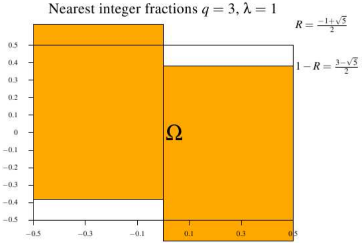
$$a_n = (y_n) \quad \text{and} \quad y_{n+1} = f(y_n) = y_n - a_n.$$

Natural extension of f

The natural extension of f is

$$\Omega \to \Omega; \quad (x, y) \mapsto \left(f(x), \frac{-1}{y + a_1}\right)$$

with $x = [0; a_1, \ldots]$.



(Ruelle's) Transfer operator

for the interval map $f : I \to I$ is defined as

$$\mathcal{L}_\beta h(x) = \sum_{y \in f^{-1}(x)} \left| \frac{df^{-1}(x)}{dx} \right|^\beta h(y)$$

on a suitable function space.

Theorem

Consider the Banachspace $V = C[-1,1] \cap C^\omega(-1,1)$ (with sup-norm). For $\mathrm{Re}(\beta) > 1$ the transfer operator $\mathcal{L}_\beta : V \times V \to V \times V$ is given by

$$\mathcal{L}_\beta \vec{h} = \begin{pmatrix} \sum_{n=3}^\infty h_1\big|_{2\beta} ST^n & + & \sum_{n=2}^\infty h_2\big|_{2\beta} ST^{-n} \\ \sum_{n=2}^\infty h_1\big|_{2\beta} ST^n & + & \sum_{n=3}^\infty h_2\big|_{2\beta} ST^{-n} \end{pmatrix}$$

where $\vec{h} = \binom{h_1}{h_2} \in V \times V$.

There exists a meromorphic continuation of $\mathcal{L}_\beta$ into the complex β-plane.

Let $Z(s)$ denote the Selberg ζ-function.

Main theorem

$$\det(1 - \mathcal{L}_\beta) = Z(\beta)\,\det(1 - \mathcal{K}_\beta)$$

where $\mathcal{K}_\beta$ is a simple operator with $\beta \to \det(1 - \mathcal{K}_\beta)$ has no poles and simple zeros in $\beta_{n,k} = n + \frac{2\pi i k}{\mathrm{const}}$, $n \in \mathbb{Z}_{\leq 0}$, $k \in \mathbb{Z}$.

Corollary

- *$\mathcal{L}_\beta$ has eigenvalue 1 if and only if $Z(\beta) = 0$ or $\beta = \beta_{n,k}$.*
- *$\mathcal{L}_\beta$ has unbounded eigenvalues for $\beta \to \beta_0$ if and only if $Z(\beta)$ has a pole at $\beta = \beta_0$.*
- *At $\beta = 0, -1, -2, \ldots$ $\mathcal{L}_\beta$ has eigenvalue 1 of the same order as the zero of $Z(\beta) + 1$.*

Lemma

Let $\vec{h} = \binom{h_1}{h_2}$ be an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue 1.
There exists a $g \in C^\omega(-r-1, r)$, $r = \frac{1+\sqrt{5}}{2}$, such that g restricted to $[-1,1]$ is h_1, $g\big|T^{-1}$ restricted to $[-1,1]$ is h_2.
Moreover g satisfies the relation

$$g = g\big|_{2\beta} \sum_{n=3}^{\infty} ST^n + g\big|_{2\beta} \sum_{n=2}^{\infty} T^{-1}ST^{-n}$$

and on $(-r, r)$ the 4-term equation

$$g\big|_{2\beta}(1 + ST^2) = g\big|_{2\beta}(T^{-1} + T^{-1}ST^{-2}).$$

If g satisfies the first relation then $\vec{h}(x) = \binom{g(x)}{g(x-1)}$ is again an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue 1.

Theorem (R.W. Bruggeman, M)

Eigenfunctions of $\mathcal{L}_\beta$, $2\beta \notin \mathbb{Z}_{\leq 2}$, with eigenvalue 1 give rise to Lewis-Zagier periodfunctions.

Geodesic flow on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$

$\downarrow$

Cross section & Return map — Continued fraction

$\downarrow$ — $\downarrow$

Interval map (Hurwitz) $\longleftarrow$ Reduction theory

$\downarrow$

Transfer operator $\longleftrightarrow$ Periodfunctions

$\updownarrow$ — $\uparrow$

Selberg ζ-function

$\searrow$ — $\swarrow$

Maass wave forms

Hecke triangle Group G_5

$G_q = \langle S, T \rangle$ such that $(ST)^5 = 1$.

- A realization is $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & \lambda_5 \\ 0 & 1 \end{pmatrix}$ with $\lambda_5 = 2\cos\left(\frac{\pi}{5}\right)$.
- G_5 is a non-arithmetic group.

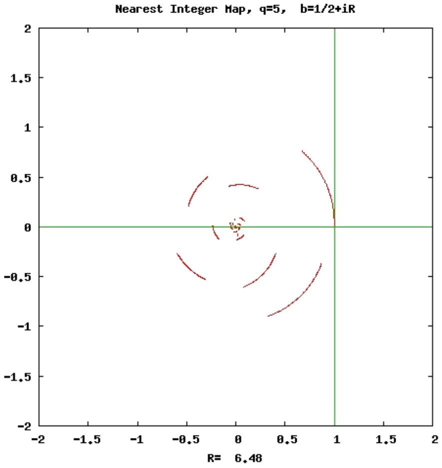
Instead of the Hurwitz continued fractions we use the

Nakada continued fractions $x = a_0\lambda_5 + \cfrac{-1}{a_1\lambda_5 + \cfrac{-1}{a_2\lambda_5 + \cfrac{-1}{a_3\lambda_5 + \ldots}}}$.

- $\rightsquigarrow$ We have a return map.
- $\rightsquigarrow$ We have an interval map.
- $\rightsquigarrow$ We have a suitable reduction theory.

$\rightsquigarrow$ *We can construct a Transfer operator $\mathcal{L}_s$ for the geodesic flow $G_5 \backslash \mathfrak{H}$.*

Spectrum of the transfer operator for $R \in [6, 14]$ and $q = 5$:



Movie