Minisymposium 10

The use of proof theory in mathematics

Leiter des Symposiums:

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The objective is to present the developments that are taking place in the fields of proof mining, exhibiting the constructive content of classical proofs, formalisation of proofs, program extraction from proofs, and the like. The stress is on the proof-theoretic methods that have been used to improve on mathematical results by, for instance, enriching them systematically with algorithms and effective bounds. Examples are the formal approach to commutative algebra performed by Coquand and Lombardi, the monotone functional interpretation used by Kohlenbach in functional analysis, and the refined A-translation applied by Berger, Buchholz, and Schwichtenberg.

Donnerstag, 21. September

Zeichensaal, Mathematisches Institut, Wegelerstr. 10

15:00 – 15:50 **Henri Lombardi** *Besançon, France()*
The Elimination of Prime Ideals

16:00 – 16:50 **Ulrich Kohlenbach** *(Darmstadt)*
Logical Metatheorems and their Use in Functional Analysis and Hyperbolic Geometry

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17:00 – 17:50 **Bas Spitters** *(Nijmegen, The Netherlands)*
Observational Integration Theory with Applications to Riesz Spaces

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Freitag, 22. September

Zeichensaal, Mathematisches Institut, Wegelerstr. 10

15:00 – 15:50 **Ulrich Berger** *(Swansea, UK)*
Program Extraction from Proofs: Theory and Practice

16:00 – 16:50 **Helmut Schwichtenberg** *(München)*
Logic for Computable Functionals and their Approximations

17:00 – 17:50 **Thomas Streicher** *(Darmstadt)*
Shoenfield = Gödel after Krivine

Vortragsauszüge

Henri Lombardi *(Besançon, France)*

The Elimination of Prime Ideals

Contrarily to André Weil, who wanted to "eliminate the elimination", i.e. eliminate computations, we think that it is important to use abstract settings in order to make computations. Unfortunately the mathematicians who invented the abstract objects did not always tell us which "concrete objects" they started from. One reason is that they had a direct intuition of these objects. Without being able to explain the mystery, we nevertheless propose concrete objects as supports of abstract ones. Here we try to understand how prime ideals can be eliminated from abstract proofs in order to obtain constructive proofs using concrete substitutes for prime ideals.
For short, when a generic prime ideal is used in order to get an algebraic result, we can often understand this machinery as a way to prove that a certain ring is trivial. The proof is ad absurdum: if the ring were not trivial, then a prime ideal would exist in this ring. There are two variants: first, if the ring were not trivial, then a maximal prime ideal would exist in this ring; secondly, if the ring were not trivial, then a minimal prime ideal would exist in this ring. We will explain the computations that are involved when deciphering several classical uses of prime ideals.

Ulrich Kohlenbach *(Darmstadt)*

Logical Metatheorems and their Use in Functional Analysis and Hyperbolic Geometry

In recent years logical metatheorems have been developed which guarantee the extractability of effective strongly uniform bounds from large classes of proofs in functional analysis and hyperbolic geometry. "Strongly uniform" refers to the fact that the bounds are independent from parameters in abstract metric, hyperbolic, CAT(0) or normed spaces as long as some local bounds on certain metric distances between these parameters are given.
We will present some recent applications in metric fixed point theory where this has led to effective uniformity results which not even ineffectively were known before. We also give a new extension of the previously known metatheorems by a powerful "nonstandard" uniform bounded principle and indicate its use.

Bas Spitters *(Nijmegen, The Netherlands)*
Observational Integration Theory with Applications to Riesz Spaces

In this talk I will present a constructive theory of integration. It illustrates the general theme of developing mathematics observationally, connecting ideas by Kolmogorov, von Neumann and Segal on the one hand and point-free (also known as formal) topology on the other. This provides a nice illustration how ideas from logic (proof theory) can be used to obtain mathematical results. As an example I will show how to mechanically remove the axiom of choice from a proof in Riesz space (vector lattice) theory, thus obtaining an elementary proof and a more general result.

Ulrich Berger *(Swansea, UK)*
Program Extraction from Proofs: Theory and Practice

This talk will give an overview of various techniques for extracting computational content from formal proofs emphasising the gap between pure methods that work in principle and refined techniques that can be applied to nontrivial examples with practically useful results.

Helmut Schwichtenberg *(München)*
Logic for Computable Functionals and their Approximations

An attempt is made to develop a constructive theory of formal neighborhoods for continuous functionals, in a direct and intuitive style. Guided by abstract domain theory, we consider a more concrete and (in the case of finitary free algebras) finitary theory of representations. As a framework for this we use Scott's information systems.

Thomas Streicher *(Darmstadt)*
Shoenfield = Gödel after Krivine

In the 1960s J. Shoenfield came up with a functional interpretation $(-)^S$ of Peano arithmetic (PA). Recently, G. Mints raised the question whether one can express $(-)^S$ as

$(A^K)^D$ where D is Gödel's Dialectica interpretation and $(-)^K$ is an appropriately chosen negative translation.
We present such a translation $(-)^K$ going back to J.-L. Krivine and elaborated by B. Reus and T. Streicher, and prove that if

$$A^S \equiv \forall u \exists x A_S(u,x) \quad \text{and} \quad (A^K)^D \equiv \exists f \forall u A_D^K(f,u)\,,$$

then $A_D^K(f,u)$ and $A_S(u,f(u))$ are provably equivalent in HA_ω.
The content of this talk is joint work with Ulrich Kohlenbach.