Structure of Hecke algebras arising from types

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Abstract

Let G denote a connected reductive group over a non-archimedean local field F of residue characteristic p, and let $\mathcal{C}$ denote an algebraically closed field of characteristic $\ell \neq p$. If ρ is an irreducible smooth $\mathcal{C}$-representation of a compact, open subgroup K of $G(F)$, then the pair (K, ρ) gives rise to a Hecke algebra $\mathcal{H}(G(F), (K, \rho))$. For a large class of pairs (K, ρ), we show that $\mathcal{H}(G(F), (K, \rho))$ is a semi-direct product of an affine Hecke algebra with explicit parameters with a twisted group algebra, and that it is isomorphic to $\mathcal{H}(G^0(F), (K^0, \rho^0))$ for some reductive subgroup $G^0 \subset G$ with compact, open subgroup K^0 and depth-zero representation ρ^0 of K^0.

The class of pairs that we consider includes all depth-zero types, and we recover as a special case of our results the depth-zero Hecke algebra description of Morris. In a second paper, we will show that our class also contains all of the types constructed by Kim and Yu, and hence we obtain as a corollary that arbitrary Bernstein blocks are equivalent to depth-zero Bernstein blocks under minor tameness assumptions.

The pairs to which our results apply are described in an axiomatic way so that the results can be applied to other constructions of types by only verifying that the relevant axioms are satisfied. The Hecke algebra isomorphisms are given in an explicit manner and are support preserving.

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Contents

1 Introduction

The category of all smooth complex representations of a p-adic group G decomposes as a product of indecomposable, full subcategories, called Bernstein blocks, each of which is equivalent to modules over a Hecke algebra under minor tameness assumptions. Therefore knowing the explicit structure of these Hecke algebras and their modules yields an understanding of the category of smooth representations. The famous example of the Iwahori–Hecke algebra has already been described in the 1960s, and the structure of the above Hecke algebras for GL_n have been known since the 1990s, and all of them played an important role in representation theory. However, comparatively little has been known about the structure of these Hecke algebras in the general setting above. In this paper, we provide an explicit description of a large class of such Hecke algebras as a semi-direct product of an affine Hecke algebra with a twisted group algebra. The Hecke algebras that we treat are endomorphism-valued functions on the p-adic group that transform on an appropriate compact, open subgroup K via a sufficiently nice irreducible representation ρ. By [FOAM], these Hecke algebras include among others the prior-mentioned Hecke algebras of Bernstein blocks that exist under minor tameness assumptions. Moreover, we obtain an isomorphisms between Hecke algebras attached to our general pairs (K, ρ) with Hecke algebras attached to much simpler pairs of reductive subgroups of G, e.g., pairs (K^0, ρ^0) with ρ^0 a depth-zero representation. In the special case of Bernstein blocks, this provides an isomorphism between a Bernstein block of positive-depth and a depth-zero Bernstein block. The Hecke algebra of the latter was essentially already known by Morris [Mor93], while the positive-depth Hecke algebras for general p-adic groups remained a mystery until now. To be precise, Morris' work only provides the Hecke algebras attached to (potentially non-singleton) finite products of depth-zero Bernstein blocks, but we also obtain a description of the Hecke algebras for single Bernstein blocks in the present paper.

The general axiomatic set-up of this present paper allows for the possibility of applying the results to, for example, Hecke algebras attached to Bernstein blocks arising from other (including future) constructions of pairs (K, ρ) that do not rely on the above minor tameness assumptions. Moreover, we allow arbitrary algebraically closed fields of characteristic different from p as our coefficents.

1.1 Overview of the main results

To explain our results in more detail, let F denote a non-archimedean local field, G a connected reductive group over F, and $\mathcal{C}$ an algebraically closed field of characteristic different from p.

To a pair consisting of a compact, open subgroup K of $G(F)$ and an irreducible smooth $\mathcal{C}$-representation (ρ, V_ρ) of K, we attach the Hecke algebra $\mathcal{H}(G(F),(K,\rho))$ of compactly supported functions $\varphi : G(F) \to \mathrm{End}_{\mathcal{C}}(V_\rho)$ that satisfy $\varphi(k_1 g k_2) = \rho(k_1) \circ \varphi(g) \circ \rho(k_2)$ for all $k_1, k_2 \in K$ and $g \in G(F)$. The algebra structure arises from the convolution recalled in Section 2.2. These algebras are closely related to smooth representations of $G(F)$ as follows. The category of smooth $\mathcal{C}$-representations decomposes into a product of indecomposable, full subcategories. In the case that $\mathcal{C} = \mathbb{C}$, one calls these subcategories *Bernstein blocks*, and under additional mild tameness assumptions, this decomposition takes the form

$$\mathrm{Rep}(G(F)) = \prod_{(K,\rho)} \mathrm{Rep}^{(K,\rho)}(G(F)),$$

where the product is taken over appropriate pairs (K,ρ) such that each block $\mathrm{Rep}^{(K,\rho)}(G(F))$ is equivalent (via an explicit equivalence recalled in Section 4.6) to the category of unital right modules over $\mathcal{H}(G(F),(K,\rho))$:

$$\mathrm{Rep}^{(K,\rho)}(G(F)) \simeq \text{Mod-}\mathcal{H}(G(F),(K,\rho))$$

([Ber84, BK98, Yu01, KY17, Fin21a, Fin21b]). The pairs (K,ρ) in the above Bernstein decomposition were constructed by Kim and Yu ([KY17, Fin21a]) and are a special case of what Bushnell and Kutzko ([BK98]) called $\mathfrak{s}$*-types*.[1]

The pairs (K,ρ) that we consider in this paper are described in terms of four axioms, Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, that distill the key properties of the $\mathfrak{s}$-types constructed by Kim and Yu that are used to prove our main results below about the structure of the attached Hecke algebra $\mathcal{H}(G(F),(K,\rho))$. Working with general pairs that satisfy only these four axioms means that our results will be applicable in a much broader setting than just for the $\mathfrak{s}$-types constructed by Kim and Yu. This is expected to include among others future constructions of $\mathfrak{s}$-types in the non-tame setting.

We now fix a pair (K,ρ) that satisfies our axioms. The first main result, Theorem 3.10.10, describes the structure of $\mathcal{H}(G(F),(K,\rho))$ via an explicit isomorphism

$$\mathcal{H}(G(F),(K,\rho)) \xrightarrow{\sim} \mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}(W(\rho_M)_{\mathrm{aff}}, q), \tag{1.1.1}$$

where $W(\rho_M)_{\mathrm{aff}}$ is an affine Weyl group that is a normal subgroup of a larger symmetry group $\Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$, $q \colon S \longrightarrow \mathcal{C}^\times$ is a parameter function on a set S of simple reflections generating $W(\rho_M)_{\mathrm{aff}}$, $\mathcal{H}(W(\rho_M)_{\mathrm{aff}}, q)$ denotes the corresponding affine Hecke algebra, and $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}]$ denotes the group algebra of $\Omega(\rho_M)$ twisted by a 2-cocycle $\mu^{\mathcal{T}}$. Both of these latter algebras, and the meaning of their semi-direct product, are recalled in Notation 3.10.8.

The second main result (see Theorems 4.4.8 and 4.4.11 and Corollary 4.5.2) is the existence of a full-rank reductive subgroup $G^0 \subseteq G$ and an irreducible representation ρ^0 of $K^0 \coloneqq K \cap G^0(F)$ with $\rho = \rho^0 \otimes \kappa$ for some smooth representation κ, where we view ρ^0 also as a representation of K via an inflation map described in Axiom 4.2.1(2), such that we obtain an explicit, support-preserving isomorphism of Hecke algebras

$$\mathcal{H}(G^0(F),(K^0,\rho^0)) \xrightarrow{\sim} \mathcal{H}(G(F),(K,\rho)). \tag{1.1.2}$$

[1] For the experts, in this introduction, we use "$\mathfrak{s}$-type" to mean "a type corresponding to a single Bernstein block". The $\mathfrak{s}$ does not refer to a specific block.

In the case where the pairs (K,ρ) and (K^0,ρ^0) are both $\mathfrak{s}$-types, this leads to an equivalence of categories (Theorem 4.6.3):

$$\mathrm{Rep}^{(K,\rho)}(G(F)) \xrightarrow{\sim} \mathrm{Rep}^{(K^0,\rho^0)}(G^0(F)). \tag{1.1.3}$$

The possibilities for the triple (G^0, K^0, ρ^0) are described in an axiomatic way, via Axioms 4.1.2, 4.2.1, and 4.3.1. In the case of the $\mathfrak{s}$-types constructed by Kim and Yu, one can always choose a triple (G^0, K^0, ρ^0) with ρ^0 of depth zero. In fact, in this case, one option for the triple is already part of the input of the construction by Kim and Yu, twisted by a quadratic character introduced by [FKS23]. Since under minor tameness assumptions every Bernstein block has an $\mathfrak{s}$-type of the form constructed by Kim and Yu, this allows one to reduce a plethora of problems about representations of p-adic groups to the depth-zero setting.

While we have chosen an axiomatic treatment distilling the key properties that are really necessary for our proofs so that the results can be applied to a variety of different constructions of types, including future ones, and different settings, including coefficients of positive characteristic, we also show in the short Section 5 that all depth-zero $\mathfrak{s}$-types as well as the depth-zero types corresponding to parahoric subgroups satisfy the axioms. Thus our first main result in that case recovers and generalizes a result of Morris ([Mor93]). The proof that all $\mathfrak{s}$-types constructed by Kim and Yu satisfy our axioms is deferred to [FOAM] as it involves the extension of the quadratic twist introduced by [FKS23] to a group of representatives for the whole support of the Hecke algebra and a careful analysis of the Heisenberg–Weil representations. This uses a different style of arguments and is of independent interest.

1.2 Some prior and related work

While our works are the first to provide an explicit description of Hecke algebras attached to such a broad class of generalizations of types of arbitrary reductive groups, which include in particular types for all Bernstein blocks if $\mathcal{C} = \mathbb{C}$, the group G splits over a tamely ramified extension and p does not divide the order of the absolute Weyl group of G, mathematicians have studied and used such Hecke algebras since the 1960s and obtained results about their structure in special cases.

A famous example of the Hecke algebras that we consider is the Iwahori–Hecke algebra attached to the Iwahori subgroup and the trivial complex representation thereof. The Iwahori–Hecke algebra was described in 1965 by Iwahori and Matsumoto for adjoint, split semisimple groups ([IM65]) and their work was fundamental for further developments in the area.

If G is split and semisimple, and one replaces the trivial character of the Iwahori subgroup I by a depth-zero character χ, then Goldstein [Gol90] computed the Hecke algebras attached to (I,χ), showing that such an algebra is isomorphic to the Iwahori–Hecke algebra of a smaller group. In a vast generalization, Morris ([Mor93]) described in 1993 all the Hecke algebras attached to pairs (K,ρ) where K is a parahoric subgroup of $G(F)$ and ρ is an irreducible, cuspidal representation of the quotient of K by its pro-p unipotent radical, i.e., (K,ρ) is a depth-zero type for a finite union of Bernstein blocks by the work of Moy and Prasad ([MP94, MP96]) and, independently, of Morris ([Mor99]).

Knowing the Hecke algebras of types of Bernstein blocks allows one to study the representations in the Bernstein block via modules of the corresponding Hecke algebra. An example of this approach is Lusztig's famous work ([Lus95, Lus02]) in which he classified all unipotent representations of adjoint simple algebraic groups that split over an unramified extension by classifying the representations of the corresponding Hecke algebras using their explicit structure. Unipotent representations are a special class of depth-zero representations, and Lusztig's work provides an important case of an explicit local Langlands correspondence. The restriction to adjoint groups was later removed by Solleveld ([Sol23]), again using the explicit structure of the affine Hecke algebras attached to types of Bernstein blocks consisting of unipotent representations.

Beyond depth-zero representations, building on the work of a lot of mathematicians on special cases over several decades, Bushnell and Kutzko ([BK93a]) provided a description of all of the Hecke algebras attached to types for all Bernstein blocks for the group GL_n in parallel to constructing these types. Their description of the Hecke algebras played an important role for the construction of the types themselves as well as for proving the exhaustiveness of their construction of supercuspidal representations. Using results from Bushnell and Kutzko on supercuspidal representations of $\mathrm{SL}(n)$ ([BK93b, BK94]), Goldberg and Roche ([GR02, GR05]) provided a complete collection of types for all Bernstein blocks of $\mathrm{SL}(n)$ and described the structure of the resulting Hecke algebras. Sécherre and Stevens ([SS08]) achieved the same for all inner forms of $\mathrm{GL}(n)$, and Miyauchi and Stevens ([MS14]) provided types for all classical groups assuming that $p \neq 2$ and described the Hecke algebras corresponding to types associated to maximal proper Levi subgroups. For general split reductive groups, Roche ([Roc98]) described types and the corresponding Hecke algebras for all principal series Bernstein blocks under mild hypotheses on p.

For reductive groups G that split over a tamely ramified field extension, Kim and Yu ([KY17]) provided a construction of types based on Yu's construction of supercuspidal representations ([Yu01, Fin21a]) that provides types for all Bernstein blocks if p does not divide the order of the absolute Weyl group of G ([Fin21b]). These types are a special case of the setting in which the results of the present paper apply, as we show in [FOAM].

Recently, Solleveld ([Sol22]) has taken a different approach to understanding Bernstein blocks of arbitrary connected reductive groups by studying modules over the endomorphism algebras $\mathrm{End}_{G(F)}(\Pi)$ of a progenerator Π of a Bernstein block. Under an assumption of the existence of a nice 2-cocycle ([Sol22, Corollary 9.4]), he proved that the category of finite-dimensional right modules over $\mathrm{End}_{G(F)}(\Pi)$ is equivalent to the category of finite-dimensional right modules over a twisted affine Hecke algebra (but these algebras are not isomorphic). Solleveld also studied the endomorphism algebra of a smaller progenerator Π_1 of a Bernstein block. Under the assumption that the supercuspidal representation of the supercuspidal support of the Bernstein block is multiplicity free when restricted to the subgroup of compact elements, which he calls "Working hypothesis 10.2" and which is often but not always satisfied, Solleveld described the endomorphism algebras $\mathrm{End}_{G(F)}(\Pi_1)$, which under minor tameness assumptions is isomorphic to the Hecke algebra $\mathcal{H}(G(F),(K,\rho))$ attached to an $\mathfrak{s}$-type (K,ρ) that we consider in this paper. Solleveld described $\mathrm{End}_{G(F)}(\Pi_1)$ as a "twisted" semi-direct product of (a more general version of) an affine Hecke algebra and a twisted group algebra. While this description looks on first glance similar to ours, a key difference is that the cocycle $\mu^{\mathcal{T}}$ of our twisted group algebra is $\mathcal{C}^\times$-valued and our twisted group algebra is a subalgebra of the Hecke algebra $\mathcal{H}(G(F),(K,\rho))$. Solleveld's 2-cocycle in contrast takes values in his affine Hecke algebra and therefore the resulting twisted group algebra is a not a subalgebra in general and the multiplication structure is more complicated than a usual semi-direct product. For depth-zero representations, Ohara ([Oha24a]) has shown that in the case where Solleveld's Working hypothesis 10.2 applies, the coefficients of the quadratic relations in Solleveld's affine Hecke algebras agree with our coefficients. In the case of classical groups, the above endomorphism algebras of progenerators had previously been described by Heiermann ([Hei11]).

While results about the structure of Hecke algebras, analogous to our result (1.1.1), have been achieved previously in many different situations, as described above, results analogous to (1.1.2) have to our knowledge previously only been obtained in very special cases and for GL_n ([BK93a]), principal series Bernstein blocks of split reductive groups ([Roc98]), and supercuspidal Bernstein blocks ([Oha24b]).

1.3 Sketch of the proofs of the main statements

We define the two isomorphisms (1.1.1) and (1.1.2) explicitly by defining explicit basis elements of the Hecke algebras $\mathcal{H}(G(F),(K,\rho))$ and $\mathcal{H}(G^0(F),(K^0,\rho^0))$ that will get mapped to each other and to the corresponding basis elements of $\mathcal{C}[\Omega(\rho_M),\mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}},q)$ under the isomorphisms. The difficult task consists of doing this in a way that preserves the algebra structures.

To provide a few more details, let us note that the pairs (K,ρ) and (K^0,ρ^0) come with Levi subgroups $M \subseteq G$ and $M^0 \subseteq G^0$ satisfying $M^0 \subseteq M$, which in the setting of $\mathfrak{s}$-types record Levi subgroups appearing in the supercuspidal supports of the corresponding Bernstein blocks, and with a point x_0 in the Bruhat–Tits building of G that is contained in the image of the Bruhat–Tits building of M^0. We also recall that $K^0 = K \cap G^0(F)$ and $\rho = \rho^0 \otimes \kappa$.

The bases for the Hecke algebras are indexed by K- and K^0-double cosets, respectively, that satisfy appropriate intertwining properties, and we show that these indexing sets agree and are isomorphic to a quotient $W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of a subgroup $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of the normalizer $N_{G^0}(M^0)(F)$ of M^0 in G^0 that preserves the image of x_0 in the reduced Bruhat–Tits building of M^0, see Proposition 3.4.12. This set-theoretic isomorphism endows the indexing set with a group structure, which we prove is isomorphic to a semi-direct product $\Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$ whose normal factor is an affine Weyl group, see Propositions 3.7.6 and 3.7.4.

The basis elements of the Hecke algebras $\mathcal{H}(G(F),(K,\rho))$ and $\mathcal{H}(G^0(F),(K^0,\rho^0))$ are then defined by reinterpreting the two Hecke algebras as $\mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}\rho\big)$ and $\mathrm{End}_{G^0(F)}\big(\mathrm{ind}_{K^0}^{G(F)}\rho^0\big)$, and composing two different intertwining operators. In order to define the (spaces for the) intertwining operators, we introduce a whole family of pairs $(K_x,\rho_x=\rho_x^0\otimes\kappa_x)$ and (K_x^0,ρ_x^0) for x in an open, dense subset $\mathcal{A}_{\mathrm{gen}}$ of an appropriate affine subspace $\mathcal{A}_{x_0}$ of the Bruhat–Tits building of G on which the group $\Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$ acts. The affine space underlying the affine Weyl group $W(\rho_M)_{\mathrm{aff}}$ is a quotient of $\mathcal{A}_{x_0}$, see Proposition 3.7.4. The families are set up in a way such that $(K,\rho) = (K_{x_0},\rho_{x_0})$ and $(K^0,\rho^0) = (K^0_{x_0},\rho^0_{x_0})$. We define the basis element $\Phi_w \in \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}\rho\big) \simeq \mathcal{H}(G(F),(K,\rho))$ attached to $w \in \Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$ to be the composition of the two intertwining operators

$$\Theta^{\mathrm{norm}}_{w^{-1}x_0|x_0}\colon \mathrm{ind}_{K_{x_0}}^{G(F)}(\rho_{x_0}) \to \mathrm{ind}_{K_{w^{-1}x_0}}^{G(F)}(\rho_{w^{-1}x_0}) \quad \text{and} \quad c_{w^{-1}x_0,w}\colon \mathrm{ind}_{K_{w^{-1}x_0}}^{G(F)}(\rho_{w^{-1}x_0}) \to \mathrm{ind}_{K_{x_0}}^{G(F)}(\rho_{x_0}), \tag{1.3.1}$$

where the first intertwining operator, $\Theta^{\mathrm{norm}}_{w^{-1}x_0|x_0}$, is constructed via a normalized integration, see Section 3.5, in particular Lemma 3.5.2 and Definition 3.5.9, and the second intertwining operator, $c_{w^{-1}x_0,w}$, results from choosing an element T_n in the one-dimensional $\mathcal{C}$-vector space $\mathrm{Hom}_{K_{nx_0}}({}^n\rho_{x_0}, \rho_{nx_0})$ for n a lift of w in $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ and ${}^n\rho_{x_0}$ the n-conjugate of $\rho_{x_0}=\rho$, see Definition 3.5.18

and (3.5.22) for details. The analogous definitions are made for G^0, K^0, ρ^0, adding a superscript "0" as appropriate.

In order to show that the basis elements defined for an appropriate choice of $\{T_n\}$ and $\{T_n^0\}$ lead to the desired algebra isomorphisms (1.1.1) and (1.1.2), we equip the indexing group $\Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$ with a length function $\ell_{\mathcal{K}\text{-rel}}$ that is trivial on $\Omega(\rho_M)$ and is the standard length function on the affine Weyl group $W(\rho_M)_{\mathrm{aff}}$ arising from a choice of generators S. We then have to check that the isomorphisms (1.1.1) and (1.1.2) preserve the multiplication of two basis elements Φ_w and $\Phi_{w'}$ in the case where $\ell_{\mathcal{K}\text{-rel}}(ww') = \ell_{\mathcal{K}\text{-rel}}(w) + \ell_{\mathcal{K}\text{-rel}}(w')$, which we call the *length-additive case*, and the case of multiplying Φ_s with itself for $s \in S \subset W(\rho_M)_{\mathrm{aff}}$, which we refer to as the *quadratic relations*. To achieve both cases, we analyse the two intertwining operators separately as well as their interaction.

To prove the required properties for the first intertwining operator we introduce a more general operator $\Theta_{y|x}\colon \operatorname{ind}_{K_x}^{G(F)}(\rho_x) \longrightarrow \operatorname{ind}_{K_y}^{G(F)}(\rho_y)$ that is defined for all $x, y \in \mathcal{A}_{\mathrm{gen}}$ and prove a variety of compatibility properties of these operators, see, for example, Lemma 3.5.2, Proposition 3.5.5, and Lemma 3.8.15 for more detailed statements. Moreover, a key step for proving the isomorphism (1.1.2) consists of relating the operator $\Theta_{y|x}$ defined for G directly with the corresponding operator $\Theta_{y|x}^0$ defined for G^0, which we achieve in Lemma 4.3.6. This result relies on a compatibility of the representations κ_x (recall: $\rho_x = \rho_x^0 \otimes \kappa_x$) for varying nearby points x, which is formalized in Axiom 4.3.1(5). In the setting of types constructed by Kim and Yu, this means a compatibility of various Heisenberg–Weil representations, which is proven in [FOAM, Corollary 3.6.14] and requires one to twist the initial construction of Kim and Yu by a quadratic character introduced in [FKS23].

The second family of intertwining operators $c_{w^{-1}x_0,w}$ depends on the choice of elements $\{T_n\}$, which are only determined up to scalars and which lead to a cocycle $\mu^{\mathcal{T}}$ on $\Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$, see Notation 3.6.1. We prove that we can choose the elements $\{T_n\}$ such that the only part of the resulting cocycle $\mu^{\mathcal{T}}$ that matters is its restriction to $\Omega(\rho_M)$ (see Proposition 3.10.7). This is the cocycle that appears in the twisted group algebra $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}]$. Moreover, we show that the choices $\{T_n\}$ for G and the choices $\{T_n^0\}$ for G^0 can be made in a compatible way that leads to the same cocycles and therefore to the desired Hecke algebra isomorphism (1.1.2). This step uses an extension of (the restriction of) κ to $N(\rho_M)^{\heartsuit}_{[x_0]_M}$, which is provided by Axiom 4.1.2(2). Checking this axiom in the setting of the twisted construction of Kim and Yu is one of the key results of [FOAM], see [FOAM, Proposition 4.3.4], which is based on [FOAM, Theorem 2.7.2]. Verifying the quadratic relations also requires again the compatibility of the κ_x for nearby points x mentioned above and forms one of the key steps of the present paper.

We note that we said that the pairs (K, ρ) and (K^0, ρ^0) need to each satisfy a few axioms, Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, in order for us to conclude (1.1.1), and a few additional axioms describing the compatibility between the two pairs, Axioms 4.1.2, 4.2.1, and 4.3.1, in order to achieve (1.1.2). In fact, in order to obtain (1.1.2) we require (K, ρ) to only satisfy Axiom 3.4.1 and we prove in Section 4 that the remaining axioms are automatically satisfied by knowing them for (K^0, ρ^0). In practice, checking the axioms for (K^0, ρ^0) might be significantly easier than for (K, ρ). As an example, we show in the rather short Section 5 that depth-zero $\mathfrak{s}$-types (K^0, ρ^0) satisfy Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. In [FOAM], we then apply our results from Section 4 to deduce that also all the positive-depth $\mathfrak{s}$-types (K, ρ) constructed by Kim and Yu ([KY17, Fin21a]) satisfy Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2

Our approach to proving the isomorphism (1.1.1) was heavily inspired by the work of Morris

([Mor93]) who dealt with the case where (K^0, ρ^0) is a complex, depth-zero type with K^0 being a parahoric subgroup. In particular, we use similar intertwining operators to those of Morris. However, to prove that the resulting basis elements of the Hecke algebras satisfy the desired relations, Morris defines a version of the first intertwining operator $\Theta_{w^{-1}x_0|x_0}$ for w contained in a larger group than the indexing group $\Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$. Our approach in contrast consists of defining the first intertwining operator not for a larger group, but, instead, of defining intertwining operators $\Theta_{y|x}$ for all $x, y \in \mathcal{A}_{\mathrm{gen}}$, i.e., also for points x and y that do not lie in the $\Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$-orbit of x_0. Although we only need the intertwining operator $\Theta_{w^{-1}x_0|x_0}$ to define the basis element Φ_w for $w \in \Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$, these general intertwining operators $\Phi_{y|x}$ allow us to prove the desired properties about Φ_w in Sections 3.5 and 3.8. We believe that this approach makes our arguments look cleaner, and at the same time allows us to treat a more general setting than the one that Morris dealt with. And it provides us with the set-up in which we can prove our second main result, (1.1.2).

1.4 Structure of the paper

We have tried to state all assumptions that apply in each subsection at the beginning of the subsection so that it is easy to see what assumptions are in place where. Moreover, in the main results and in propositions that get used later within the paper, we have repeated the assumptions that are in place to provide clarity for the reader and allow for easier backtracking through the results and to make it easier to follow our proofs. We have also tried to state all of the axioms as early as possible in each subsection so that a reader interested in simply knowing what axioms they need to verify to apply the results can reach the statements of the axioms as quickly as possible. A list of axioms is provided on page 82. We have also created a list of notation (page 85).

We give a brief overview of the following sections of this paper.

In §2.1 we fix some notation, and in §2.2, we recall the definition of a Hecke algebra as a convolution algebra of functions and its relation with the second interpretation as an endomorphism algebra, because we will use both points of view.

In Section 3, we prove the structure theorem (1.1.1) for the Hecke algebra associated to a pair (K, ρ) that satisfies Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2 defined there. For this, we first construct an explicit vector-space basis $\{\varphi_w\}$ of $\mathcal{H}(G(F), (K, \rho))$ indexed by the group $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ under the assumptions of Axioms 3.4.1 and 3.4.3 as follows. In §3.4, we introduce the family (K_x, ρ_x) mentioned above indexed by $\mathcal{A}_{\mathrm{gen}}$. In §3.5, for $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$, we define the intertwining operators $\Theta^{\mathrm{norm}}_{w^{-1}x_0|x_0}$ and $c_{w^{-1}x_0,w}$ (see (1.3.1)) and define the element Φ_w of $\mathrm{End}_{G(F)}\big(\mathrm{ind}_{K_{x_0}}^{G(F)}(\rho_{x_0})\big)$ as their composition. We then define the basis element $\varphi_w \in \mathcal{H}(G(F), (K, \rho))$ to be the element corresponding to Φ_w via the isomorphism $\mathcal{H}(G(F), (K, \rho)) \simeq \mathrm{End}_{G(F)}\big(\mathrm{ind}_{K_{x_0}}^{G(F)}(\rho_{x_0})\big)$ given in §2.2. Based on discussions about the intertwining operators in §3.5, we investigate the relations involving the elements φ_w in the length-additive case. In §3.7, we prove that the indexing group $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ is isomorphic to the semi-direct product $\Omega(\rho_M) \ltimes W(\rho_M)_{\mathrm{aff}}$ mentioned above under the assumption of Axiom 3.7.1. In §3.8, we introduce Axiom 3.8.2, and under this axiom, we prove quadratic relations for the elements φ_s corresponding to simple reflections $s \in W(\rho_M)_{\mathrm{aff}}$. When the characteristic ℓ of $\mathcal{C}$ is zero or a banal prime, the coefficients of the quadratic relations are explicitly calculated in §3.9. In §3.10, we prove the structure theorem (1.1.1) by proving appropriate braid relations and combining them with the discussions from previous subsections. In §3.11, we assume that the coefficient field $\mathcal{C}$ admits a nontrivial involution. Under this assumption, we prove that we can take the isomorphism in (1.1.1) such that it preserves anti-involutions on both sides. In §3.12, we classify the support-preserving isomorphisms in (1.1.1).

In Section 4, we prove the Hecke algebra isomorphism (1.1.2) for a pair (K, ρ) consisting of a compact, open subgroup K of $G(F)$ and an irreducible representation ρ of K and a similar pair (K^0, ρ^0) for a reductive subgroup G^0 of G that are related according to Axioms 4.1.2, 4.2.1, and 4.3.1. We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2 for the pair (K^0, ρ^0) but only assume Axiom 3.4.1 for the pair (K, ρ). We prove in §4.4 that the other axioms for (K, ρ) automatically follow in this case. Thus, by applying the results in Section 3 to the pairs (K^0, ρ^0) and (K, ρ), we obtain the descriptions of the Hecke algebras as semi-direct products of affine Hecke algebras with twisted group algebras. The results of §4.3 imply that the affine Weyl groups and the subgroups of length-zero elements for (K^0, ρ^0) and (K, ρ) agree. In §4.4, we prove the Hecke algebra isomorphism (1.1.2). We have already shown that each of $\mathcal{H}(G(F), (K, \rho))$ and $\mathcal{H}(G^0(F), (K^0, \rho^0))$ is isomorphic to a semi-direct product of an affine Hecke algebra (where the underlying affine reflection groups are the same) with a twisted group algebra (where the underlying groups are the same). We complete the argument by showing that, if certain choices are made carefully, then the parameters of the two affine Hecke algebras match up, as well as the two cocycles that implement the twisting of the two group algebras. In §4.5, under the assumption that $\mathcal{C}$ admits a nontrivial involution, we prove that we can take the isomorphism in (1.1.2) such that it preserves anti-involutions on both sides. As an application, in §4.6, we prove the equivalence of Bernstein blocks in (1.1.3). Moreover, we also prove that when restricted to irreducible objects, the equivalence of Bernstein blocks preserves temperedness, and preserves the Plancherel measure on the tempered dual up to an explicit constant factor (see Theorem 4.6.4).

In Section 5, we determine the structure of all Hecke algebras arising from depth-zero pairs (K, ρ). We review the construction of such pairs in §5.1, and describe the system of hyperplanes that applies to this case in §5.2. In §5.3, we obtain the structure of the Hecke algebras (Theorem 5.3.6) by showing that depth-zero pairs satisfy those axioms from Section 3 necessary to allow us to apply Theorem 3.10.10.

1.5 Guidance for the reader

If a reader is interested in our results only in the case of types as constructed by Kim and Yu, we suggest the following approach to the paper: Start with [FOAM, Section 4] and read until [FOAM, Section 4.2] to learn the definitions of the objects G, M, x_0, $\mathfrak{H}$, K_M, ρ_M, K_x, $K_{x,+}$, ρ_x and G^0, M^0, K_{M^0}, ρ_{M^0}, K_x^0, $K_{x,+}^0$, ρ_x^0. Afterwards, we encourage the reader to read Section 3 and 4 of the present paper but to replace in their mind the abstract objects G, M, x_0, $\mathfrak{H}$, K_M, ρ_M, K_x, $K_{x,+}$, ρ_x and G^0, M^0, K_{M^0}, ρ_{M^0}, K_x^0, $K_{x,+}^0$, ρ_x^0 with the explicit objects introduced in [FOAM, Section 4]. We have provided a lot of cross-references between [FOAM, Section 4] and Sections 3 and 4 of this paper to ease this approach.

For a reader who is interested in our general results and who is already familiar with either the construction of Kim and Yu or depth-zero types, we have added after the introduction of each abstract object an explanation or reference regarding what these objects are in these specific settings, and we provide references to where the axioms are proven in these settings. Such a reader might also benefit from skimming or reading [FOAM, Section 4] until [FOAM, Section 4.2] (for the types as constructed by Kim and Yu) and/or Section 5 until Section 5.2 (for the depth-zero setting) in parallel to reading Sections 3 and 4 of the present paper.

For a reader using the axiomatic set-up to prove similar results in other settings, we have provided a complete list of axioms on page 82 with references to where the axioms can be found. Moreover, we have tried to state the axioms as early as possible in their subsections to allow the reader to receive the desired information, i.e., the axioms to check, as quickly as possible.

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2 Preliminaries

2.1 Notation

Let F be a non-archimedean local field endowed with a discrete valuation ord on $F^\times$ with the value group $\mathbb{Z}$. For a finite field extension E of F, we also write ord for the unique extension of this valuation to $E^\times$. We write $\mathcal{O}_E$ for the ring of integers of E, and let $\mathcal{O} = \mathcal{O}_F$. Let p denote the characteristic of the residue field $\mathfrak{f}$ of F.

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. Let $\mathcal{C}$ denote an algebraically closed field of characteristic $\ell \neq p$. Except when otherwise indicated, all representations below are on vector spaces over $\mathcal{C}$. We fix a square root $p^{1/2}$ of p in $\mathcal{C}$. For an element $c = p^n \in \mathcal{C}$ with $n \in \mathbb{Z}$, we write $c^{1/2} := \left(p^{1/2}\right)^n$.

For a torus S that is defined and splits over F, we let $X^*(S)$ denote the character group of S and $X_*(S)$ the cocharacter group of S.

For a connected reductive group G defined over F, we write $Z(G)$ for the center of G and A_G for the maximal split torus in $Z(G)$. For a reductive subgroup M of G, we denote by $N_G(M)$ (resp. $Z_G(M)$) the normalizer (resp. centralizer) of M in G.

We denote by $\mathcal{B}(G, F)$ the enlarged Bruhat–Tits building of G, and for a maximal split torus S of G, we denote by $\mathcal{A}(G, S, F)$ the apartment of S in $\mathcal{B}(G, F)$. We also write $\mathcal{B}^{\text{red}}(G, F)$ for the reduced building of G and $\mathcal{A}^{\text{red}}(G, S, F)$ for the apartment of S in $\mathcal{B}^{\text{red}}(G, F)$. For $x \in \mathcal{B}(G, F)$, we denote by $[x]_G$ the image of x in $\mathcal{B}^{\text{red}}(G, F)$. We might also write $[x]$ instead of $[x]_G$ if the group G is clear from the context. We write $G(F)_x$ and $G(F)_{[x]}$ for the stabilizers of x and $[x]$ in $G(F)$. We denote by $G(F)_{x,0}$ the group of $\mathcal{O}$-points of the connected parahoric group scheme of G associated to the point x, and by $G(F)_{x,0+}$ the pro-p radical of $G(F)_{x,0}$. The reductive quotient of the special fiber of the connected parahoric group scheme above will be denoted by G_x.

Thus, $\mathsf{G}_x(\mathfrak{f}) = G(F)_{x,0}/G(F)_{x,0+}$. (We will use this font convention more widely. Thus, for a Levi subgroup M of G, and $x \in \mathcal{B}(M,F)$, we have the $\mathfrak{f}$-group M_x.)

If P is a parabolic subgroup of G, then let U_P denote the unipotent radical of P. We define the set $\mathcal{U}(M)$ as

$$\mathcal{U}(M) = \{U_P \mid P \subseteq G \text{ is parabolic with Levi factor } M\}.$$

Suppose that K is an open subgroup of a locally profinite group H. For a smooth representation (ρ, V_ρ) of K, we denote by

$$\left(\operatorname{ind}_K^H(\rho), \operatorname{ind}_K^H(V_\rho)\right)$$

the compactly induced representation of (ρ, V_ρ). Here, we realize $\operatorname{ind}_K^H(\rho)$ as the right regular representation on

$$\operatorname{ind}_K^H(V_\rho) = \{f\colon H \to V_\rho\colon \text{compactly supported modulo } K \mid f(kh) = \rho(k)(f(h))\,(k \in K, h \in H)\}.$$

Suppose that K is a subgroup of a group H and $h \in H$. We denote hKh^{-1} by ${}^hK$. If ρ is a representation of K, we write ${}^h\rho$ for the representation $x \mapsto \rho(h^{-1}xh)$ of ${}^hK$. If an element $h \in H$ satisfies

$$\operatorname{Hom}_{K \cap {}^hK}({}^h\rho, \rho) \neq \{0\},$$

we say that h *intertwines* ρ. We write

$$I_H(\rho) = \{h \in H \mid h \text{ intertwines } \rho\}.$$

We also write

$$N_H(K) = \left\{h \in H \mid {}^hK = K\right\} \quad \text{and} \quad N_H(\rho) = \left\{h \in N_H(K) \mid {}^h\rho \simeq \rho\right\}.$$

For a representation (ρ, V_ρ) of a group H, we identify ρ with its representation space V_ρ by abuse of notation. For any vector space V, we write id_V for the identity map on V.

Throughout the paper, we let G be a connected reductive group defined over F.

2.2 Hecke algebras and endomorphism algebras

Let K be a compact, open subgroup of $G(F)$ and let (ρ, V_ρ) be an irreducible smooth representation of K. We recall the definition of the Hecke algebra associated to the pair (K,ρ). Let $\mathcal{H}(G(F),(K,\rho))$ denote the space of compactly supported functions

$$\varphi\colon G(F) \to \operatorname{End}_{\mathcal{C}}(V_\rho)$$

satisfying

$$\varphi(k_1 g k_2) = \rho(k_1) \circ \varphi(g) \circ \rho(k_2)$$

for all $k_1, k_2 \in K$ and $g \in G(F)$. We put an associative $\mathcal{C}$-algebra structure on $\mathcal{H}(G(F),(K,\rho))$, from now on called the *Hecke algebra associated to the pair* (K,ρ), as follows. Suppose that φ_1 and φ_2 are functions in $\mathcal{H}(G(F),(K,\rho))$. We define the function $\varphi_1 * \varphi_2 \in \mathcal{H}(G(F),(K,\rho))$ as

$$(\varphi_1 * \varphi_2)(g) = \sum_{h \in G(F)/K} \varphi_1(h) \circ \varphi_2(h^{-1}g)$$

for $g \in G(F)$ (see [Vig96, §8.6]). We note that since φ_1 is compactly supported, the sum on the right-hand side is a finite sum. We also note that when the characteristic ℓ of $\mathcal{C}$ is zero, then this multiplication is equivalent to the standard convolution operation

$$(\varphi_1 * \varphi_2)(g) = \int_{G(F)} \varphi_1(h) \circ \varphi_2(h^{-1}g)\, dh,$$

where dh denotes a Haar measure on $G(F)$ which is chosen so that K has measure one. (However, the isomorphism class of $\mathcal{H}(G(F), (K, \rho))$ does not depend on the choice of measure.) If the group K is clear from the context, we drop it from the notation and write $\mathcal{H}(G(F), \rho)$ for $\mathcal{H}(G(F), (K, \rho))$.

Remark 2.2.1. The definition of $\mathcal{H}(G(F), \rho)$ above is different from the definition of $\mathcal{H}(G(F), \rho)$ in [BK98, Section 2]. More precisely, our $\mathcal{H}(G(F), \rho)$ is denoted by $\mathcal{H}(G(F), \rho^\vee)$ in [BK98, Section 2], where $\rho^\vee$ denotes the contragredient representation of ρ. We also note that our $\mathcal{H}(G(F), \rho)$ is written as $\check{\mathcal{H}}(G(F), \rho)$ in [KY17, Section 8] and [Yu01, Section 17].

For $\varphi \in \mathcal{H}(G(F), \rho)$, we write $\mathrm{supp}(\varphi)$ for the support of φ. For $g \in G(F)$, $\varphi \in \mathcal{H}(G(F), \rho)$, and $k \in K \cap {}^gK$, we have

$$\rho(k) \circ \varphi(g) = \varphi(kg) = \varphi(g(g^{-1}kg)) = \varphi(g) \circ \rho(g^{-1}kg) = \varphi(g) \circ {}^g\rho(k),$$

and hence

$$\varphi(g) \in \mathrm{Hom}_{K \cap {}^gK}({}^g\rho, \rho).$$

Therefore we obtain that $\mathrm{supp}(\varphi) \subset I_{G(F)}(\rho)$ for any $\varphi \in \mathcal{H}(G(F), \rho)$. For $g \in G(F)$, we define the subspace $\mathcal{H}(G(F), \rho)_g$ of $\mathcal{H}(G(F), \rho)$ as

$$\mathcal{H}(G(F), \rho)_g = \{\varphi \in \mathcal{H}(G(F), \rho) \mid \mathrm{supp}(\varphi) \subset KgK\}.$$

The subspace $\mathcal{H}(G(F), \rho)_g$ is zero if $g \notin I_{G(F)}(\rho)$, and it depends only on the double coset KgK, and not on the choice of the coset representative g. As a vector space, we have

$$\mathcal{H}(G(F), \rho) = \bigoplus_{g \in K \backslash I_{G(F)}(\rho)/K} \mathcal{H}(G(F), \rho)_g. \tag{2.2.2}$$

According to [Vig96, Section 8.5], there exists an isomorphism of $\mathcal{C}$-algebras

$$\mathcal{H}(G(F), \rho) \xrightarrow{\sim} \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big). \tag{2.2.3}$$

We write the isomorphism above explicitly. For $v \in V_\rho$, we define $f_v \in \mathrm{ind}_K^{G(F)}(V_\rho)$ as

$$f_v(g) = \begin{cases} \rho(g)(v) & (g \in K), \\ 0 & (\text{otherwise}). \end{cases}$$

Then for $\Phi \in \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)$, the corresponding element $\varphi \in \mathcal{H}(G(F), \rho)$ is defined by

$$\varphi(g)(v) = (\Phi(f_v))\,(g) \tag{2.2.4}$$

for $v \in V_\rho$ and $g \in G(F)$. Conversely, for $\varphi \in \mathcal{H}(G(F), \rho)$, the corresponding element

$$\Phi \in \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)$$

is defined by

$$(\Phi(f))\,(g) = \sum_{h \in G(F)/K} \varphi(h)\left(f(h^{-1}g)\right)$$

for $f \in \mathrm{ind}_K^{G(F)}(V_\rho)$ and $g \in G(F)$.

For a subgroup K' of $G(F)$ containing K, we identify the compactly induced representation $\mathrm{ind}_K^{K'}(\rho)$ with the K'-subrepresentation of $\mathrm{ind}_K^{G(F)}(\rho)$ on the space

$$\left\{ f \in \mathrm{ind}_K^{G(F)}(V_\rho) \,\middle|\, \mathrm{supp}(f) \subset K' \right\},$$

where $\mathrm{supp}(f)$ denotes the support of f. More generally, for a subset K' of $G(F)$ such that $K \cdot K' = K'$, we define a subspace $\mathrm{ind}_K^{K'}(V_\rho)$ of $\mathrm{ind}_K^{G(F)}(V_\rho)$ as

$$\mathrm{ind}_K^{K'}(V_\rho) = \left\{ f \in \mathrm{ind}_K^{G(F)}(V_\rho) \,\middle|\, \mathrm{supp}(f) \subset K' \right\}.$$

In particular, we regard (ρ, V_ρ) as a K-subrepresentation of $(\mathrm{ind}_K^{G(F)}(\rho), \mathrm{ind}_K^{G(F)}(V_\rho))$ via the isomorphism

$$\rho \xrightarrow{\sim} \mathrm{ind}_K^K(\rho)$$

defined via

$$V_\rho \ni v \longmapsto f_v \in \mathrm{ind}_K^K(V_\rho).$$

Lemma 2.2.5. *Let $g \in K \backslash I_{G(F)}(\rho)/K$. Let $\Phi \in \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)$ and $\varphi \in \mathcal{H}(G(F), \rho)$ correspond to each other via the isomorphism in* (2.2.3)*. Then the element φ is supported on KgK if and only if*

$$\Phi\,(V_\rho) \subset \mathrm{ind}_K^{KgK}(V_\rho).$$

Proof. According to the construction of the isomorphism in (2.2.3), we have

$$\varphi(h)(v) = (\Phi(f_v))\,(h)$$

for all $h \in G$. Hence, the function φ is supported on KgK if and only if for any $v \in V_\rho$, we have

$$\mathrm{supp}\,(\Phi(f_v)) \subset KgK,$$

that is, $\Phi\,(f_v) \in \mathrm{ind}_K^{KgK}(V_\rho)$. □

For a subgroup K' of $G(F)$ containing K, we define a subalgebra $\mathcal{H}(K', \rho)$ of $\mathcal{H}(G(F), \rho)$ as

$$\mathcal{H}(K', \rho) = \left\{ \varphi \in \mathcal{H}(G(F), \rho) \mid \mathrm{supp}(\varphi) \subset K' \right\}.$$

According to Lemma 2.2.5, the subalgebra $\mathcal{H}(K', \rho)$ corresponds to the subalgebra $\mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)_{K'}$ of $\mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)$ defined as

$$\begin{aligned}\mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)_{K'} &= \Big\{\Phi \in \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big) \mid \Phi\left(V_\rho\right) \subset \mathrm{ind}_K^{K'}(V_\rho)\Big\} \\ &= \Big\{\Phi \in \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big) \mid \Phi\left(\mathrm{ind}_K^{K'}(V_\rho)\right) \subset \mathrm{ind}_K^{K'}(V_\rho)\Big\}\end{aligned}$$

via the isomorphism in (2.2.3). Similar to (2.2.3), we also have an isomorphism

$$\mathcal{H}(K', \rho) \xrightarrow{\sim} \mathrm{End}_{K'}\big(\mathrm{ind}_K^{K'}(\rho)\big), \tag{2.2.6}$$

and one can check easily that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}(K', \rho) & \xrightarrow{(2.2.3)} & \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)_{K'} \\ \mathrm{id}\downarrow & \circlearrowleft & \downarrow\mathrm{res} \\ \mathcal{H}(K', \rho) & \xrightarrow{(2.2.6)} & \mathrm{End}_{K'}\big(\mathrm{ind}_K^{K'}(\rho)\big), \end{array} \tag{2.2.7}$$

where res denotes the restriction map defined by

$$\Phi \longmapsto \Phi|_{\mathrm{ind}_K^{K'}(V_\rho)}.$$

In particular, the restriction map is an isomorphism. We regard $\mathrm{End}_{K'}\big(\mathrm{ind}_K^{K'}(\rho)\big)$ as a subalgebra of $\mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)$ via the inverse of this isomorphism.

We will also need a slightly more general setup. For this, let H be a locally profinite group and let K_1, K_2 be open subgroups of H. Let ρ_1, resp., ρ_2, be a smooth representation of K_1, resp., K_2. For an open subgroup K' of H containing K_1 and K_2, we define a subspace $\mathrm{Hom}_H\left(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\right)_{K'}$ of $\mathrm{Hom}_H\left(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\right)$ by

$$\begin{aligned}\mathrm{Hom}_H\left(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\right)_{K'} &= \Big\{\Phi \in \mathrm{Hom}_H\left(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\right) \Big| \Phi\left(V_{\rho_1}\right) \subset \mathrm{ind}_{K_2}^{K'}(V_{\rho_2})\Big\} \\ &= \Big\{\Phi \in \mathrm{Hom}_H\Big(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\Big) \Big| \Phi\left(\mathrm{ind}_{K_1}^{K'}(V_{\rho_1})\right) \subset \mathrm{ind}_{K_2}^{K'}(V_{\rho_2})\Big\}.\end{aligned}$$

Lemma 2.2.8. *The restriction map* $\Phi \mapsto \Phi|_{\mathrm{ind}_{K_1}^{K'}(V_{\rho_1})}$ *gives an isomorphism*

$$\mathrm{Hom}_H\left(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\right)_{K'} \xrightarrow{\sim} \mathrm{Hom}_{K'}\left(\mathrm{ind}_{K_1}^{K'}(\rho_1), \mathrm{ind}_{K_2}^{K'}(\rho_2)\right).$$

Proof. Combining Frobenius reciprocity with the transitivity of compact induction, we obtain an isomorphism

$$\begin{aligned}\mathrm{Hom}_H\big(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\big) &\simeq \mathrm{Hom}_H\big(\mathrm{ind}_{K'}^H(\mathrm{ind}_{K_1}^{K'}(\rho_1)), \mathrm{ind}_{K_2}^H(\rho_2)\big) \\ &\simeq \mathrm{Hom}_{K'}\big(\mathrm{ind}_{K_1}^{K'}(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)|_{K'}\big).\end{aligned}$$

Under this isomorphism the subspace $\mathrm{Hom}_H\big(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\big)_{K'}$ of $\mathrm{Hom}_H\big(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\big)$ corresponds to the subspace $\mathrm{Hom}_{K'}\big(\mathrm{ind}_{K_1}^{K'}(\rho_1), \mathrm{ind}_{K_2}^{K'}(\rho_2)\big)$ of $\mathrm{Hom}_{K'}\big(\mathrm{ind}_{K_1}^{K'}(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)|_{K'}\big)$, and the induced isomorphism $\mathrm{Hom}_H\big(\mathrm{ind}_{K_1}^H(\rho_1), \mathrm{ind}_{K_2}^H(\rho_2)\big)_{K'} \xrightarrow{\sim} \mathrm{Hom}_{K'}\big(\mathrm{ind}_{K_1}^{K'}(\rho_1), \mathrm{ind}_{K_2}^{K'}(\rho_2)\big)$ agrees with the restriction map. □

3 Structure of the Hecke algebra

We recall that G is a connected reductive group defined over F. In this section, we will provide a description of the Hecke algebra attached to an appropriate compact, open subgroup $K_{x_0} \subset G$ and an appropriate representation $(\rho_{x_0}, V_{\rho_{x_0}})$ of K_{x_0} as a semi-direct product of an affine Hecke algebra and a twisted group algebra, see Theorem 3.10.10. We describe the pairs (K_{x_0}, ρ_{x_0}) that we consider through several axioms introduced in this section. These are Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2 below. The pairs that we consider include among others the case where (K_{x_0}, ρ_{x_0}) is a type for a single Bernstein block as constructed by Kim and Yu ([KY17, Fin21a]) or a depth-zero type attached to either a single Bernstein block or the case where K_{x_0} is a parahoric subgroup, which is the case studied previously by Morris ([Mor93]).

Readers interested in depth-zero types might find it useful to first read Sections 5.1 and 5.2, readers interested in the types constructed by Kim and Yu might find it helpful to first read Sections 4.1 and 4.2 in [FOAM], in order to keep these special cases as examples in mind when reading the axiomatic set-up below. Those interested only in the set-up in [FOAM] might even completely replace the below axiomatic objects by the explicit objects introduced in [FOAM, Section 4] in their head.

3.1 The affine space

In this subsection, we will introduce an affine subspace $\mathcal{A}_{x_0}$ of $\mathcal{B}(G, F)$, which will be used to index pairs of compact, open subgroups K_x and irreducible smooth representations ρ_x of K_x below. More precisely, we will consider a family $\{(K_x, \rho_x)\}_{x \in \mathcal{A}_{\mathrm{gen}}}$ of compact, open subgroups K_x of $G(F)$ and irreducible smooth representations ρ_x of K_x indexed by an appropriate subset $\mathcal{A}_{\mathrm{gen}}$ of generic points in $\mathcal{A}_{x_0}$ (see Sections 3.2 and 3.4). The family $\{(K_x, \rho_x)\}_{x \in \mathcal{A}_{\mathrm{gen}}}$ will be used to define basis elements of a Hecke algebra (see Definition 3.5.23). In Proposition 3.7.4 below, we will also define an affine root system $\Gamma(\rho_M)$ on a quotient $\mathcal{A}_{\mathcal{K}\text{-rel}}$ of $\mathcal{A}_{x_0}$, whose affine Weyl group underlies the affine Hecke algebra appearing in our description of the Hecke algebra attached to (K_{x_0}, ρ_{x_0}) in Theorem 3.10.10.

To define the affine space $\mathcal{A}_{x_0}$, we fix a Levi subgroup M of G. If we want to describe the Hecke algebra attached to a type for a Bernstein block of a $\mathbb{C}$-representation, then M is a Levi subgroup of the supercuspidal support of the Bernstein block. We recall that we write A_M for the maximal split torus of the center $Z(M)$ of M. Let $\mathcal{B}(M, F) \hookrightarrow \mathcal{B}(G, F)$ be an admissible embedding of enlarged Bruhat–Tits buildings in the sense of [KP23, §14.2], which exists by [KP23, Section 9.7.5]. We regard $\mathcal{B}(M, F)$ as a subset of $\mathcal{B}(G, F)$ via this embedding. Let $x_0 \in \mathcal{B}(M, F)$. We define the subset $\mathcal{A}_{x_0}$ of $\mathcal{B}(M, F)$ by

$$\mathcal{A}_{x_0} = x_0 + (X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}).$$

More precisely, $x_0 + (X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R})$ is an affine subspace of every apartment containing x_0 (since A_M is contained in every maximal split torus of M), and the image of this affine space in $\mathcal{B}(M, F)$ is independent of the choice of apartment. We fix an $N_G(M)(F)$-invariant inner product $(\ ,\ )_M$ on $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. (Such an inner product exists, because the action of $N_G(M)(F)$ factors through a finite group.) Hence, the space $\mathcal{A}_{x_0}$ is a Euclidean space with the vector space of translations $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. We define the subset $N_G(M)(F)_{[x_0]_M}$ of $G(F)$ by

$$N_G(M)(F)_{[x_0]_M} = \{n \in N_G(M)(F) \mid n x_0 \in \mathcal{A}_{x_0}\}.$$

Lemma 3.1.1. *Let S be a maximal split torus of M such that $x_0 \in \mathcal{A}(G,S,F)$. Then we have*

$$N_G(M)(F)_{[x_0]_M} = \big(N_G(M)(F)_{[x_0]_M} \cap N_G(S)(F)\big) \cdot M(F)_{x_0,0}.$$

Proof. Let $n \in N_G(M)(F)_{[x_0]_M}$. Since $x_0 \in \mathcal{A}(G,S,F)$, and $nx_0 \in \mathcal{A}_{x_0} \subset \mathcal{A}(G,S,F)$, we have $x_0 \in \mathcal{A}(G,S,F) \cap \mathcal{A}(G, n^{-1}Sn, F)$. Hence, there exists an element $m \in M(F)_{x_0,0}$ such that $m(n^{-1}Sn)m^{-1} = S$. Thus, we obtain that

$$n = nm^{-1} \cdot m \in \big(N_G(M)(F)_{[x_0]_M} \cap N_G(S)(F)\big) \cdot M(F)_{x_0,0}.$$

□

Lemma 3.1.2. *Let $n \in N_G(M)(F)_{[x_0]_M}$. Then we have $nx \in \mathcal{A}_{x_0}$ for all $x \in \mathcal{A}_{x_0}$.*

Proof. Let S be a maximal split torus of M such that $x_0 \in \mathcal{A}(G,S,F)$. According to Lemma 3.1.1, we have

$$N_G(M)(F)_{[x_0]_M} = \big(N_G(M)(F)_{[x_0]_M} \cap N_G(S)(F)\big) \cdot M(F)_{x_0,0}.$$

Since $mx = x$ for all $m \in M(F)_{x_0,0}$ and $x \in \mathcal{A}_{x_0}$, it suffices to show that $nx \in \mathcal{A}_{x_0}$ for all

$$n \in N_G(M)(F)_{[x_0]_M} \cap N_G(S)(F)$$

and $x \in \mathcal{A}_{x_0}$. We write $x = x_0 + a$ for some

$$a \in X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} \subset X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}.$$

According to [KP23, Proposition 6.2.4], we have

$$nx = n(x_0 + a) = nx_0 + (Dn)a,$$

where Dn denotes the image of n in the finite Weyl group $N_G(S)(F)/Z_G(S)(F)$, which acts on the vector space $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Since n normalizes M and $a \in X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$, we have

$$(Dn)a \in X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Thus, the assumption $n \in N_G(M)(F)_{[x_0]_M}$ implies that

$$nx = nx_0 + (Dn)a \subset \mathcal{A}_{x_0} + (X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}) = \mathcal{A}_{x_0}.$$

□

For later application it will be useful to state the result for an arbitrary point $x_0' \in \mathcal{A}_{x_0}$.

Corollary 3.1.3. *Let $x_0' \in \mathcal{A}_{x_0}$. If an element $n \in N_G(M)(F)$ satisfies $nx_0' \in \mathcal{A}_{x_0}$, then we have $nx \in \mathcal{A}_{x_0}$ for all $x \in \mathcal{A}_{x_0}$.*

Proof. This follows from Lemma 3.1.2 by replacing x_0 by x_0'. □

According to Lemma 3.1.2, $N_G(M)(F)_{[x_0]_M}$ is a subgroup of $G(F)$, and the action of $G(F)$ on $\mathcal{B}(G,F)$ induces an action of $N_G(M)(F)_{[x_0]_M}$ on $\mathcal{A}_{x_0}$. For $n \in N_G(M)(F)_{[x_0]_M}$ and a subset X of $\mathcal{A}_{x_0}$, we write $n(X) = \{nx \mid x \in X\}$.

3.2 Affine hyperplanes

Let $\mathfrak{H}$ be a possibly empty, locally finite set of affine hyperplanes in $\mathcal{A}_{x_0}$ that do not contain x_0. The complement of these hyperplanes in $\mathcal{A}_{x_0}$ decomposes into connected components and the compact open subgroup K_x that we will attach each point x in the complement will be constant on those components. The image on the quotient $\mathcal{A}_{\mathcal{K}\text{-rel}}$ of $\mathcal{A}_{x_0}$ of an appropriate subset $\mathfrak{H}_{\mathcal{K}\text{-rel}} \subseteq \mathfrak{H}$ of $\mathcal{K}$-relevant hyperplanes will form the set of vanishing hyperplanes of the affine root system $\Gamma(\rho_M)$ mentioned above in Section 3.1 (see Definition 3.5.6 and Proposition 3.7.4). In the setting of depth-zero types discussed in Section 5, the set of affine hyperplanes $\mathfrak{H}$ that we want to consider is described in §5.2. In the setting of positive-depth types as constructed by Kim and Yu, the set of affine hyperplanes $\mathfrak{H}$ that we want to consider is described in [FOAM, Section 4.2]. We define the subset $\mathcal{A}_{\text{gen}}$ of *generic points* of $\mathcal{A}_{x_0}$ by

$$\mathcal{A}_{\text{gen}} = \mathcal{A}_{x_0} \smallsetminus \left(\bigcup_{H \in \mathfrak{H}} H \right).$$

For $x, y \in \mathcal{A}_{\text{gen}}$, we define the subset $\mathfrak{H}_{x,y}$ of $\mathfrak{H}$ by

$$\mathfrak{H}_{x,y} = \{H \in \mathfrak{H} \mid x \text{ and } y \text{ are on opposite sides of } H\}.$$

Since $\mathfrak{H}$ is locally finite, we have $\#\mathfrak{H}_{x,y} < \infty$. We write $d(x,y) = \#\mathfrak{H}_{x,y}$.

Lemma 3.2.1. *Let $x, y, z \in \mathcal{A}_{\text{gen}}$. Then we have*

$$d(x,y) + d(y,z) \geq d(x,z).$$

Moreover, the following conditions are equivalent:

(a) $$d(x,y) + d(y,z) = d(x,z).$$
(b) $$\mathfrak{H}_{x,y},\ \mathfrak{H}_{y,z} \subset \mathfrak{H}_{x,z}.$$
(c) $$\mathfrak{H}_{x,y} \cap \mathfrak{H}_{y,z} = \emptyset.$$

Proof. For any $H \in \mathfrak{H}_{x,z}$, exactly one of the following occurs:

- We have $H \in \mathfrak{H}_{x,y}$.
- We have $H \in \mathfrak{H}_{y,z}$.

Hence, we have $d(x,y) + d(y,z) \geq d(x,z)$, and equality holds if and only if $\mathfrak{H}_{x,y}, \mathfrak{H}_{y,z} \subset \mathfrak{H}_{x,z}$. We will prove that conditions (b) and (c) are equivalent. Suppose that $\mathfrak{H}_{x,y}, \mathfrak{H}_{y,z} \subset \mathfrak{H}_{x,z}$. Then, for any $H \in \mathfrak{H}_{x,y}$, we also have $H \in \mathfrak{H}_{x,z}$. Since the points x and y are on opposite sides of H, and the points x and z are on opposite sides of H, we obtain that the points y and z are on the same side of H. Hence, we have $H \notin \mathfrak{H}_{y,z}$. Thus, we conclude that $\mathfrak{H}_{x,y} \cap \mathfrak{H}_{y,z} = \emptyset$. On the other hand, suppose that $\mathfrak{H}_{x,y} \cap \mathfrak{H}_{y,z} = \emptyset$. Then, for any $H \in \mathfrak{H}_{x,y}$, the points y and z are on the same side of H. Hence, the points x and z are on opposite sides of H, that is $H \in \mathfrak{H}_{x,z}$. Thus, we conclude that $\mathfrak{H}_{x,y} \subset \mathfrak{H}_{x,z}$. Similarly, we can prove that $\mathfrak{H}_{y,z} \subset \mathfrak{H}_{x,z}$. □

3.3 Quasi-G-covers

Let K_M be a compact, open subgroup of $M(F)_{x_0}$ and let (ρ_M, V_{ρ_M}) be an irreducible smooth representation of K_M. For example, the pairs (K_M, ρ_M) considered in [FOAM, Section 4] include the supercuspidal types constructed by Yu ([Yu01]) twisted by a quadratic character that arises from the work of [FKS23].

Notation 3.3.1. We denote by $N(\rho_M)_{[x_0]_M}$ the subgroup of $N_G(M)(F)_{[x_0]_M}$ given by

$$N(\rho_M)_{[x_0]_M} = N_{G(F)}(\rho_M) \cap N_G(M)(F)_{[x_0]_M}.$$

Definition 3.3.2. Let K (resp. K_M) be a compact, open subgroup of $G(F)$ (resp. $M(F)$) and let ρ (resp. ρ_M) be an irreducible smooth representation of K (resp. K_M). We say that (K, ρ) is a *quasi-G-cover* of the pair (K_M, ρ_M) if for every parabolic subgroup $P \subseteq G$ with Levi factor M, we have that the pair (K, ρ) is *decomposed* with respect to (M, P) (in the sense of [BK98, Definition 6.1]). Equivalently, for every $U \in \mathcal{U}(M)$ the following conditions are satisfied:

(1) We have the decomposition

$$K = (K \cap U(F)) \cdot (K \cap M(F)) \cdot \left(K \cap \overline{U}(F)\right).$$

(2) We have $K_M = K \cap M(F)$, the restriction of ρ to K_M is ρ_M, and the restriction of ρ to the groups $K \cap U(F)$ and $K \cap \overline{U}(F)$ is trivial.

Remark 3.3.3. Let M be a Levi subgroup of G and $U \in \mathcal{U}(M)$. Then, according to [Bor91, Proposition 14.21 (iii)], the product map

$$U(F) \times M(F) \times \overline{U}(F) \to G(F)$$

is a homeomorphism onto an open subset of $G(F)$. Hence, Condition (1) of the definition of a quasi-G-cover is equivalent to the condition that the product map

$$(K \cap U(F)) \times (K \cap M(F)) \times \left(K \cap \overline{U}(F)\right) \to K$$

is a homeomorphism of topological spaces. In particular, any element of K can be written uniquely as a product of three elements in $K \cap U(F), K \cap M(F)$, and $K \cap \overline{U}(F)$, respectively.

Remark 3.3.4. If (K, ρ) is a G-cover of (K_M, ρ_M) in the sense of [BK98, Definition 8.1], then (K, ρ) is a quasi-G-cover of (K_M, ρ_M) by definition.

Quasi-G-covers will allow us to compare intertwiners of representations of compact, open subgroups of $G(F)$ with intertwiners of representations of compact, open subgroups of $M(F)$. More precisely, we have the following lemma that will be used to study the support of Hecke algebras.

Lemma 3.3.5. *Let K_1 and K_2 be compact, open subgroups of $G(F)$, and let $K_{M,1}$ and $K_{M,2}$ be compact, open subgroups of $M(F)$. Let ρ_1, ρ_2, $\rho_{M,1}$, and $\rho_{M,2}$ be irreducible smooth representations of K_1, K_2, $K_{M,1}$, and $K_{M,2}$, respectively. Suppose that the pair (K_i, ρ_i) is a quasi-G-cover of the pair $(K_{M,i}, \rho_{M,i})$ for $i = 1, 2$. Then we have*

$$\mathrm{Hom}_{K_1 \cap K_2}(\rho_1, \rho_2) = \mathrm{Hom}_{K_{M,1} \cap K_{M,2}}(\rho_{M,1}, \rho_{M,2}).$$

Proof. We fix $U \in \mathcal{U}(M)$. Since (K_i, ρ_i) is a quasi-G-cover of $(K_{M,i}, \rho_{M,i})$, we have

$$K_i = (K_i \cap U(F)) \cdot K_{M,i} \cdot \left(K_i \cap \overline{U}(F)\right),$$

the representation ρ_i is trivial on $K_i \cap U(F)$ and $K_i \cap \overline{U}(F)$, and we have $\rho_i|_{K_{M,i}} = \rho_{M,i}$ for $i = 1, 2$. Thus we obtain

$$\begin{aligned} \mathrm{Hom}_{K_1 \cap K_2}(\rho_1, \rho_2) &= \mathrm{Hom}_{(K_1 \cap K_2 \cap U(F)) \cdot \left(K_{M,1} \cap K_{M,2}\right) \cdot \left(K_1 \cap K_2 \cap \overline{U}(F)\right)}(\rho_1, \rho_2) \\ &= \mathrm{Hom}_{K_{M,1} \cap K_{M,2}}(\rho_1, \rho_2) \\ &= \mathrm{Hom}_{K_{M,1} \cap K_{M,2}}(\rho_{M,1}, \rho_{M,2}). \end{aligned}$$

□

Corollary 3.3.6. *Let K be a compact, open subgroup of $G(F)$ and ρ be an irreducible smooth representation of K. Suppose that (K, ρ) is a quasi-G-cover of (K_M, ρ_M). Then we have*

$$N(\rho_M)_{[x_0]_M} \subset I_{G(F)}(\rho).$$

Proof. Let $n \in N(\rho_M)_{[x_0]_M}$. Since the pair (K, ρ) is a quasi-G-cover of (K_M, ρ_M) and n normalizes the group M, the pair $({}^nK, {}^n\rho)$ is a quasi-G-cover of $({}^nK_M, {}^n\rho_M)$. Then Lemma 3.3.5 implies that

$$\mathrm{Hom}_{K \cap {}^nK}({}^n\rho, \rho) = \mathrm{Hom}_{K_M \cap {}^nK_M}({}^n\rho_M, \rho_M).$$

Since $n \in N_{G(F)}(\rho_M)$, the right-hand side is non-trivial. Hence, we obtain that the left-hand side is also non-trivial, that is, $n \in I_{G(F)}(\rho)$. □

3.4 Family of quasi-G-covers and a group structure on the Hecke algebra support

We will define and study basis elements of the Hecke algebra attached to a quasi-G-cover of (K_M, ρ_M) below. To do this, we study not just one quasi-G-cover, but rather a family of quasi-G-covers of (K_M, ρ_M) indexed by $\mathcal{A}_{\mathrm{gen}}$ with some additional data and properties. More precisely, consider a family

$$\mathcal{K} = \{(K_x, K_{x,+}, (\rho_x, V_{\rho_x}))\}_{x \in \mathcal{A}_{\mathrm{gen}}},$$

where each K_x is a compact, open subgroup of $G(F)$, each $K_{x,+}$ is a normal, open subgroup of K_x, and each (ρ_x, V_{ρ_x}) is an irreducible smooth representation of K_x. We will refer to such a family as a *family of quasi-G-cover-candidates*. Let $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ be a subgroup of $N(\rho_M)_{[x_0]_M}$ containing $A_M(F)$. Eventually (starting with Corollary 3.4.14) we will assume that $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ is sufficiently large, more precisely large enough to satisfy Axiom 3.4.3 below. For now we first only assume that $\mathcal{K}$ and $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ satisfy the following:

Axiom 3.4.1.

(1) The restriction of ρ_x to $K_{x,+}$ is θ_x-isotypic for some character θ_x of $K_{x,+}$.

(2) The action of $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ preserves the set $\mathfrak{H}$. In particular, $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ stabilizes the the subset $\mathcal{A}_{\mathrm{gen}}$ of $\mathcal{A}_{x_0}$.

(3) For every $x \in \mathcal{A}_{\mathrm{gen}}$ and $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$, we have

$$K_{nx} = nK_xn^{-1} \qquad \text{and} \qquad K_{nx,+} = nK_{x,+}n^{-1}.$$

(4) For every $x \in \mathcal{A}_{\text{gen}}$ and for all $U \in \mathcal{U}(M)$, we have

   (a) the pair (K_x, ρ_x) is a quasi-G-cover of (K_M, ρ_M),
   (b) $K_x = K_M \cdot K_{x,+}$,
   (c) $K_{x,+} = (K_{x,+} \cap U(F)) \cdot (K_{x,+} \cap M(F)) \cdot \left(K_{x,+} \cap \overline{U}(F)\right)$.

   Moreover, the group $K_{x,+} \cap M(F)$ is independent of the point $x \in \mathcal{A}_{\text{gen}}$.

(5) For $x, y, z \in \mathcal{A}_{\text{gen}}$ such that
$$d(x,y) + d(y,z) = d(x,z),$$
   there exists $U \in \mathcal{U}(M)$ such that
$$K_x \cap U(F) \subseteq K_y \cap U(F) \subseteq K_z \cap U(F)$$
   and
$$K_z \cap \overline{U}(F) \subseteq K_y \cap \overline{U}(F) \subseteq K_x \cap \overline{U}(F).$$
   In particular, for any $x, y \in \mathcal{A}_{\text{gen}}$, there exists $U \in \mathcal{U}(M)$ such that
$$K_x \cap U(F) \subseteq K_y \cap U(F) \quad \text{and} \quad K_y \cap \overline{U}(F) \subseteq K_x \cap \overline{U}(F).$$

Once we assume this axiom for a family of quasi-G-cover-candidates $\mathcal{K}$, we also refer to $\mathcal{K}$ as a *family of quasi-G-covers.*

Remark 3.4.2. The last paragraph of Section 5.1, see p. 77, summarizes the families that we consider in Section 5, and the last paragraph of [FOAM, Section 4.1], see p. 43 in [FOAM], summarizes the families that we consider in [FOAM]. These cases include the types for single depth-zero Bernstein blocks and those for positive-depth Bernstein blocks as constructed by Kim and Yu (twisted by a quadratic character following Fintzen, Kaletha and Spice), respectively. Axiom 3.4.1 for the families considered in Section 5 is vertified in Lemma 5.3.1, and Axiom 3.4.1 for the families considered in [FOAM] is verified in [FOAM, Lemma 4.3.2].

Recall that the support of the Hecke algebras attached to the pairs (K_x, ρ_x) for $x \in \mathcal{A}_{\text{gen}}$ is given by $I_{G(F)}(\rho_x)$. According to Corollary 3.3.6 and Axiom 3.4.1(4a), we have
$$N(\rho_M)^{\heartsuit}_{[x_0]_M} \subset N(\rho_M)_{[x_0]_M} \subset I_{G(F)}(\rho_x)$$
for all $x \in \mathcal{A}_{\text{gen}}$. Since $I_{G(F)}(\rho_x) = K_x \cdot I_{G(F)}(\rho_x) \cdot K_x$ by definition, we also have the inclusion
$$K_x \cdot N(\rho_M)^{\heartsuit}_{[x_0]_M} \cdot K_x \subset I_{G(F)}(\rho_x).$$

In order to access the support of the relevant Hecke algebras via $N(\rho_M)^{\heartsuit}_{[x_0]_M}$, which will allow us to enrich the support with a group structure in Corollary 3.4.14 and Definition 3.4.15, we will also suppose starting from Corollary 3.4.14 that this inclusion is in fact an equality, i.e., that the following axiom holds.

Axiom 3.4.3. We have
$$K_x \cdot N(\rho_M)^{\heartsuit}_{[x_0]_M} \cdot K_x = I_{G(F)}(\rho_x)$$
for all $x \in \mathcal{A}_{\text{gen}}$.

In the depth-zero setting of Section 5 this axiom holds by Proposition 5.3.2, and in the setting of [FOAM] this axiom holds by [FOAM, Proposition 4.3.5].

Remark 3.4.4. To prove Lemma 3.4.5–Lemma 3.4.13 below, we only need Axiom 3.4.1, but we believe that having all axioms in one place will help the reader find them more easily.

We record some consequences of Axiom 3.4.1 that we will use throughout the paper.

Lemma 3.4.5. *Let $x \in \mathcal{A}_{\mathrm{gen}}$. Then we have*

$$K_x \cap U(F) = K_{x,+} \cap U(F)$$

for all $U \in \mathcal{U}(M)$.

Proof. According to Axiom 3.4.1(4a), we have

$$K_x = (K_x \cap U(F)) \cdot K_M \cdot \left(K_x \cap \overline{U}(F)\right).$$

On the other hand, according to Axiom 3.4.1(4b,4c), we have

$$\begin{aligned} K_x &= K_M \cdot K_{x,+} \\ &= K_M \cdot (K_{x,+} \cap U(F)) \cdot (K_{x,+} \cap M(F)) \cdot \left(K_{x,+} \cap \overline{U}(F)\right) \\ &= (K_{x,+} \cap U(F)) \cdot K_M \cdot \left(K_{x,+} \cap \overline{U}(F)\right). \end{aligned}$$

Then the lemma follows from Remark 3.3.3. □

Lemma 3.4.6. *Let $x, y \in \mathcal{A}_{\mathrm{gen}}$. Then we have*

$$K_x \cap K_{y,+} = K_{x,+} \cap K_{y,+}.$$

Proof. We fix $U \in \mathcal{U}(M)$. By using Axiom 3.4.1(4), we obtain that

$$K_x \cap K_{y,+} = (K_x \cap K_{y,+} \cap U(F)) \cdot (K_x \cap K_{y,+} \cap M(F)) \cdot \left(K_x \cap K_{y,+} \cap \overline{U}(F)\right)$$

and

$$K_{x,+} \cap K_{y,+} = (K_{x,+} \cap K_{y,+} \cap U(F)) \cdot (K_{x,+} \cap K_{y,+} \cap M(F)) \cdot \left(K_{x,+} \cap K_{y,+} \cap \overline{U}(F)\right).$$

Then the claim follows from Lemma 3.4.5 and the fact that $K_{x,+} \cap M(F) = K_{y,+} \cap M(F)$. □

Lemma 3.4.7. *Let $x, y \in \mathcal{A}_{\mathrm{gen}}$ and $U \in \mathcal{U}(M)$ such that*

$$K_x \cap U(F) \subseteq K_y \cap U(F) \quad \text{and} \quad K_y \cap \overline{U}(F) \subseteq K_x \cap \overline{U}(F).$$

Then the inclusions $K_y \cap U(F) \subset K_{y,+} \subset K_y$ induce bijections

$$(K_y \cap U(F)) \,/\, (K_x \cap U(F)) \simeq K_{y,+}/\,(K_{x,+} \cap K_{y,+}) \simeq K_y/\,(K_x \cap K_y)\,.$$

Proof. According to Axiom 3.4.1(4a), we have

$$K_x = (K_x \cap U(F)) \cdot K_M \cdot \left(K_x \cap \overline{U}(F)\right) \quad \text{and} \quad K_y = (K_y \cap U(F)) \cdot K_M \cdot \left(K_y \cap \overline{U}(F)\right).$$

Then the assumptions of the lemma imply that

$$(K_y \cap U(F)) / (K_x \cap U(F)) \simeq K_y / (K_x \cap K_y).$$

Since $K_y \cap U(F) = K_{y,+} \cap U(F) \subset K_{y,+}$ by Lemma 3.4.5, the claim follows from Lemma 3.4.6. □

Lemma 3.4.8. *Let $x, y \in \mathcal{A}_{\text{gen}}$. Then the order of the quotient $K_y/(K_x \cap K_y)$ is a power of p. In particular, this integer is invertible in $\mathcal{C}$.*

Proof. According to Axiom 3.4.1(5), there exists $U \in \mathcal{U}(M)$ such that $K_x \cap U(F) \subseteq K_y \cap U(F)$ and $K_y \cap \overline{U}(F) \subseteq K_x \cap \overline{U}(F)$. Then the claim follows from Lemma 3.4.7 and the fact that $K_y \cap U(F)$ is a pro-p-group. □

Lemma 3.4.9. *Let $x, y \in \mathcal{A}_{\text{gen}}$. Then we have*

$$\operatorname{Hom}_{K_x \cap K_y}(\rho_x, \rho_y) \neq \{0\}.$$

In particular, we have

$$\theta_x|_{K_{x,+} \cap K_{y,+}} = \theta_y|_{K_{x,+} \cap K_{y,+}}.$$

Proof. Since (K_x, ρ_x) and (K_y, ρ_y) are quasi-G-covers of (K_M, ρ_M), Lemma 3.3.5 implies that

$$\operatorname{Hom}_{K_x \cap K_y}(\rho_x, \rho_y) = \operatorname{End}_{K_M}(\rho_M) \neq \{0\}.$$

The last claim follows from the assumption that the restriction of ρ_x to $K_{x,+}$ is θ_x-isotypic and the restriction of ρ_y to $K_{y,+}$ is θ_y-isotypic. □

Lemma 3.4.10. *Let $x \in \mathcal{A}_{\text{gen}}$ and $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$. Then the pair $(K_{nx}, {}^n\rho_x)$ is a quasi-G-cover of the pair $(K_M, {}^n\rho_M)$.*

Proof. Since the pair (K_x, ρ_x) is a quasi-G-cover of (K_M, ρ_M) and n normalizes the group M, the pair $({}^nK_x, {}^n\rho_x)$ is a quasi-G-cover of $({}^nK_M, {}^n\rho_M)$. Then the lemma follows from the facts that ${}^nK_x = K_{nx}$ and ${}^nK_M = K_M$. □

Lemma 3.4.11. *Let $x \in \mathcal{A}_{\text{gen}}$ and $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$. Then we have*

$$\theta_x(n^{-1}kn) = \theta_{nx}(k)$$

for all $k \in K_{nx,+}$.

Proof. According to Axiom 3.4.1(4a), the pair (K_{nx}, ρ_{nx}) is a quasi-G-cover of the pair (K_M, ρ_M). On the other hand, according to Lemma 3.4.10, the pair $(K_{nx}, {}^n\rho_x)$ is a quasi-G-cover of the pair $(K_M, {}^n\rho_M)$. Since $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M} \subset N_{G(F)}(\rho_M)$, Lemma 3.3.5 implies that

$$\operatorname{Hom}_{K_{nx}}({}^n\rho_x, \rho_{nx}) = \operatorname{Hom}_{K_M}({}^n\rho_M, \rho_M) \neq \{0\}.$$

Since the restriction of ${}^n\rho_x$ to $K_{nx,+}$ is ${}^n\theta_x$-isotypic, and the restriction of ρ_{nx} to $K_{nx,+}$ is θ_{nx}-isotypic, we deduce the claim. □

Axiom 3.4.1 also allows us to prove the following proposition that will be used to study the support of the Hecke algebra attached to (K_x, ρ_x).

Proposition 3.4.12. *Let $x \in \mathcal{A}_{\mathrm{gen}}$ and assume Axiom 3.4.1. Then the inclusion*

$$N(\rho_M)^{\heartsuit}_{[x_0]_M} \subset I_{G(F)}(\rho_x),$$

induces an injection

$$N(\rho_M)^{\heartsuit}_{[x_0]_M} / \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right) \to K_x \backslash I_{G(F)}(\rho_x)/K_x.$$

To prove Proposition 3.4.12, we first record the following general lemma:

Lemma 3.4.13. *Let $U \in \mathcal{U}(M)$. Then we have*

$$N_G(M)(F) \cap \overline{U}(F) \cdot U(F) = \{1\}.$$

Proof. Let $n \in N_G(M)(F)$, $u \in U(F)$, and $\overline{u} \in \overline{U}(F)$ such that $n = \overline{u}u$. For any $m \in M(F)$, we have

$$nmn^{-1} = \overline{u}umu^{-1}\overline{u}^{-1} = \overline{u}u(mu^{-1}m^{-1})(m\overline{u}^{-1}m^{-1})m.$$

We write

$$m' = (nmn^{-1})m^{-1} \in M(F) \quad \text{and} \quad u' = u(mu^{-1}m^{-1}) \in U(F).$$

Then we have

$$m'(m\overline{u}m^{-1}) = \overline{u}u'.$$

Since the natural map

$$M(F) \times \overline{U}(F) \times U(F) \to G(F)$$

is injective, we have $m' = 1$. Hence, we obtain that n commutes with any element in $M(F)$. In particular, n is contained in the centralizer of A_M in $G(F)$, that is equal to $M(F)$. Thus, we conclude that

$$n \in M(F) \cap \overline{U}(F) \cdot U(F) = \{1\}.$$

□

Proof of Proposition 3.4.12. Let $n_1, n_2 \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ and $k_1, k_2 \in K_x$ such that $n_2 = k_1 n_1 k_2$. It suffices to show that $n_1 n_2^{-1} \in K_M$. We write

$$k_2' = n_1 k_2 n_1^{-1} \in n_1 K_x n_1^{-1} = K_{n_1 x}.$$

By Axiom 3.4.1(5), there exists $U \in \mathcal{U}(M)$ such that

$$K_x \cap U(F) \subset K_{n_1 x} \cap U(F) \quad \text{and} \quad K_{n_1 x} \cap \overline{U}(F) \subset K_x \cap \overline{U}(F).$$

Then according to Axiom 3.4.1(4a), we have

$$\begin{aligned} n_2 n_1^{-1} &= k_1 n_1 k_2 n_1^{-1} \\ &= k_1 k_2' \\ &\in K_x \cdot K_{n_1 x} \\ &= \left(K_x \cap \overline{U}(F)\right) K_M \left(K_x \cap U(F)\right) \cdot K_{n_1 x} \\ &= \left(K_x \cap \overline{U}(F)\right) \cdot K_{n_1 x} \\ &= \left(K_x \cap \overline{U}(F)\right) \cdot \left(K_{n_1 x} \cap \overline{U}(F)\right) \left(K_{n_1 x} \cap U(F)\right) K_M \\ &= \left(K_x \cap \overline{U}(F)\right) \left(K_{n_1 x} \cap U(F)\right) K_M. \end{aligned}$$

Hence, there exists $k_M \in K_M$ such that

$$n_2 n_1^{-1} k_M \in N(\rho_M)^{\heartsuit}_{[x_0]_M} \cdot K_M \cap \overline{U}(F) \cdot U(F) \subset N_G(M)(F) \cap \overline{U}(F) \cdot U(F).$$

According to Lemma 3.4.13, we have $n_2 n_1^{-1} k_M = 1$. Thus, we obtain that $n_1 n_2^{-1} = k_M \in K_M$. $\square$

From now on, we suppose Axiom 3.4.3 in order to turn the injection in Proposition 3.4.12 into a bijection.

Corollary 3.4.14. *Let $x \in \mathcal{A}_{\mathrm{gen}}$ and assume Axioms 3.4.1 and 3.4.3. Then the inclusion*

$$N(\rho_M)^{\heartsuit}_{[x_0]_M} \subset I_{G(F)}(\rho_x)$$

induces a bijection

$$N(\rho_M)^{\heartsuit}_{[x_0]_M} / \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right) \simeq K_x \backslash I_{G(F)}(\rho_x)/K_x.$$

Proof. The corollary follows from Axiom 3.4.3 and Proposition 3.4.12. $\square$

Definition 3.4.15. We define the group $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ as

$$W(\rho_M)^{\heartsuit}_{[x_0]_M} = N(\rho_M)^{\heartsuit}_{[x_0]_M} / \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right).$$

Since $K_M \subset M(F)_{x_0}$ fixes every point of $\mathcal{A}_{x_0}$, the action of $N_G(M)(F)_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ induces an action of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{x_0}$.

Remark 3.4.16. Since the kernel of the action of $N_G(M)(F)_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ is the group $M(F)_{x_0}$, the action of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ is faithful if and only if

$$N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M = N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap M(F)_{x_0}.$$

In particular, if $K_M = M(F)_{x_0}$, then the action of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ is faithful.

Lemma 3.4.17. *The group $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ acts on $\mathcal{A}_{x_0}$ properly, that is, for all compact subsets C_1 and C_2 of $\mathcal{A}_{x_0}$, the set $\{w \in W(\rho_M)^{\heartsuit}_{[x_0]_M} \mid w(C_1) \cap C_2 \neq \emptyset\}$ is finite.*

Proof. Since $N(\rho_M)^{\heartsuit}_{[x_0]_M} \subset N_G(M)(F)_{[x_0]_M}$, and the quotient $M(F)_{x_0}/K_M$ is finite, it suffices to show that the group $N_G(M)(F)_{[x_0]_M}/M(F)_{x_0}$ acts properly on $\mathcal{A}_{x_0}$. Let S be a maximal split torus of M such that $x_0 \in \mathcal{A}(G, S, F)$. Let $Z_G(S)(F)_{\mathrm{cpt}}$ denote the maximal compact subgroup of $Z_G(S)(F)$. According to Lemma 3.1.1, we have

$$\begin{aligned} N_G(M)(F)_{[x_0]_M}/M(F)_{x_0} &\simeq \left(N_G(M)(F)_{[x_0]_M} \cap N_G(S)(F)\right) / \left(M(F)_{x_0} \cap N_G(S)(F)\right) \\ &\twoheadleftarrow \left(N_G(M)(F)_{[x_0]_M} \cap N_G(S)(F)\right) / Z_G(S)(F)_{\mathrm{cpt}}. \end{aligned}$$

Then the lemma follows from the fact that the Iwahori–Weyl group $N_G(S)(F)/Z_G(S)(F)_{\mathrm{cpt}}$ acts properly on the apartment $\mathcal{A}(G, S, F)$ of S. Although the fact is well known, we record the proof of it for completeness. Since the finite Weyl group $N_G(S)(F)/Z_G(S)(F)$ is finite, it suffices to show that the group $Z_G(S)(F)/Z_G(S)(F)_{\mathrm{cpt}}$ acts properly on $\mathcal{A}(G,S,F)$. According to [KP23, Proposition 6.2.4], for $z \in Z_G(S)(F)$ and $x \in \mathcal{A}(G,S,F)$, we have $zx = x + \nu(z)$, where $\nu(z)$ is the element of $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ defined by $\chi(\nu(z)) = -\operatorname{ord}(\chi(z))$ for all $\chi \in X^*(S)$. Thus, the claim follows from the fact that ν identifies the group $Z_G(S)(F)/Z_G(S)(F)_{\mathrm{cpt}}$ with a lattice in $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. $\square$

For $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$, we define $\mathcal{H}(G(F), \rho_x)_w$ to be $\mathcal{H}(G(F), \rho_x)_n$ for any representative n of w in $N(\rho_M)^{\heartsuit}_{[x_0]_M}$, i.e., the subspace of $\mathcal{H}(G(F), \rho_x)$ consisting of functions whose support is contained in $K_x n K_x$.

Proposition 3.4.18. *Let $x \in \mathcal{A}_{\mathrm{gen}}$ and assume Axioms 3.4.1 and 3.4.3. Then for each $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$, the subspace $\mathcal{H}(G(F), \rho_x)_w$ is one dimensional. Moreover, we have*

$$\mathcal{H}(G(F), \rho_x) = \bigoplus_{w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}} \mathcal{H}(G(F), \rho_x)_w$$

as a vector space.

Proof. Let $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$. We fix a lift n of w in $N(\rho_M)^{\heartsuit}_{[x_0]_M}$. Then the map $\varphi \mapsto \varphi(n)$ defines an isomorphism of vector spaces

$$\mathcal{H}(G(F), \rho_x)_w \xrightarrow{\sim} \operatorname{Hom}_{K_x \cap {}^n K_x}({}^n\rho_x, \rho_x).$$

Since (K_x, ρ_x) is a quasi-G-cover of (K_M, ρ_M), and $({}^nK_x, {}^n\rho_x)$ is a quasi-G-cover of $(K_M, {}^n\rho_M)$, Lemma 3.3.5 implies that

$$\operatorname{Hom}_{K_x \cap {}^n K_x}({}^n\rho_x, \rho_x) = \operatorname{Hom}_{K_M}({}^n\rho_M, \rho_M).$$

Since $n \in N_{G(F)}(\rho_M)$ and the representation ρ_M is irreducible, the right-hand side is one dimensional. The last claim follows from (2.2.2) and Corollary 3.4.14. $\square$

3.5 Intertwining operators and a basis of the Hecke algebra

We keep the notation from the previous subsection including the assumption of Axiom 3.4.1, but we do not assume Axiom 3.4.3 until the last corollary, Corollary 3.5.27. We will define a non-zero element $\varphi_{x,w} \in \mathcal{H}(G(F), \rho_x)_w$ for every $x \in \mathcal{A}_{\mathrm{gen}}$ and $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ below. The element $\varphi_{x,w}$ will correspond to the composition of two intertwining operators via the isomorphism in (2.2.3). The first intertwining operator

$$\Theta_{y|x} \colon \operatorname{ind}_{K_x}^{G(F)}(\rho_x) \to \operatorname{ind}_{K_y}^{G(F)}(\rho_y)$$

for $x, y \in \mathcal{A}_{\mathrm{gen}}$ is defined as follows. We note that by Lemma 3.4.9 and Axiom 3.4.1(1), for every $f \in \operatorname{ind}_{K_x}^{G(F)}(V_{\rho_x})$ and $g \in G(F)$, the element $\theta_y(k) \cdot f(k^{-1}g)$ for $k \in K_{y,+}$ only depends on the image $[k]$ of k in the quotient $K_{y,+}/(K_{x,+} \cap K_{y,+})$. For $f \in \operatorname{ind}_{K_x}^{G(F)}(V_{\rho_x})$, we define the function

$$\Theta_{y|x}(f) \colon G(F) \to V_{\rho_y}$$

by

$$\left(\Theta_{y|x}(f)\right)(g) = |K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1} \sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})} \theta_y(k)\cdot f(k^{-1}g).$$

Here, we regard $|K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|$ as an element of $\mathcal{C}$, and it is invertible by Lemmas 3.4.7 and 3.4.8.

Remark 3.5.1. If the characteristic ℓ of $\mathcal{C}$ is zero or the group $K_{y,+}$ is a pro-p-group, then we can write $\Theta_{y|x}$ as

$$\left(\Theta_{y|x}(f)\right)(g) = \int_{K_{y,+}} \theta_y(k)\cdot f(k^{-1}g)\, dk$$

for $f \in \operatorname{ind}_{K_x}^{G(F)}\left(V_{\rho_x}\right)$ and $g \in G(F)$, where dk denotes the Haar measure on $K_{y,+}$ such that the volume of $K_{y,+}$ is one. In our application of the theory to the setting of [FOAM], the group $K_{y,+}$ will be a pro-p-group, see [FOAM, (4.1.3)].

Lemma 3.5.2. *The map $\Theta_{y|x}$ defines a $G(F)$-equivariant map*

$$\Theta_{y|x}\colon \operatorname{ind}_{K_x}^{G(F)}(\rho_x) \to \operatorname{ind}_{K_y}^{G(F)}(\rho_y).$$

Proof. We will prove that $\Theta_{y|x}(f) \in \operatorname{ind}_{K_y}^{G(F)}\left(V_{\rho_y}\right)$ for all $f \in \operatorname{ind}_{K_x}^{G(F)}\left(V_{\rho_x}\right)$. Since we have $K_y = K_M \cdot K_{y,+}$ and $\rho_y|_{K_M} = \rho_M$, it suffices to show that

$$\left(\Theta_{y|x}(f)\right)(k_M g) = \rho_M(k_M)\left(\left(\Theta_{y|x}(f)\right)(g)\right)$$

and

$$\left(\Theta_{y|x}(f)\right)(k_+ g) = \rho_y(k_+)\left(\left(\Theta_{y|x}(f)\right)(g)\right)$$

for all $g \in G(F)$, $k_M \in K_M$, and $k_+ \in K_{y,+}$. Let $g \in G(F)$ and $k_M \in K_M$. Since we have $K_M \subset K_x \cap K_y$ and the group K_x (resp. K_y) normalizes $K_{x,+}$ (resp. $K_{y,+}$), the group K_M normalizes $K_{x,+}$ and $K_{y,+}$. Moreover, since K_y normalizes $K_{y,+}$ and the restriction of ρ_y to $K_{y,+}$ is θ_y-isotypic, the group K_y normalizes θ_y. Hence, we have

$$\begin{aligned}
\left(\Theta_{y|x}(f)\right)(k_M g) &= |K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1} \sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})} \theta_y(k)\cdot f(k^{-1}k_M g) \\
&= |K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1} \sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})} \theta_y(k)\cdot f(k_M k_M^{-1} k^{-1}k_M g) \\
&= |K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1} \sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})} \theta_y(k)\cdot \rho_M(k_M)\left(f(k_M^{-1} k^{-1}k_M g)\right) \\
&= \rho_M(k_M)\left(|K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1} \sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})} \theta_y(k)\cdot f(k_M^{-1} k^{-1}k_M g)\right) \\
&= \rho_M(k_M)\left(|K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1} \sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})} \theta_y(k)\cdot f((k_M^{-1} k k_M)^{-1} g)\right) \\
&= \rho_M(k_M)\left(|K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1} \sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})} \theta_y(k_M k k_M^{-1})\cdot f(k^{-1} g)\right)
\end{aligned}$$

$$= \rho_M(k_M)\left(|K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1}\sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})}\theta_y(k)\cdot f(k^{-1}g)\right)$$
$$= \rho_M(k_M)\left(\left(\Theta_{y|x}(f)\right)(g)\right).$$

Moreover, the definition of $\Theta_{y|x}$ implies that

$$\left(\Theta_{y|x}(f)\right)(k_+g) = \theta_y(k_+)\cdot\left(\Theta_{y|x}(f)\right)(g) = \rho_y(k_+)\left(\left(\Theta_{y|x}(f)\right)(g)\right)$$

for all $g \in G(F)$ and $k_+ \in K_{y,+}$.

The claim that $\Theta_{y|x}$ is $G(F)$-equivariant follows from the definition of it. □

We can also write $\Theta_{y|x}$ as follows.

Lemma 3.5.3. *Let $x, y \in \mathcal{A}_{\text{gen}}$ and $U \in \mathcal{U}(M)$ such that*

$$K_x \cap U(F) \subseteq K_y \cap U(F) \quad \text{and} \quad K_y \cap \overline{U}(F) \subseteq K_x \cap \overline{U}(F).$$

Then we have

$$\left(\Theta_{y|x}(f)\right)(g) = \int_{K_y\cap U(F)} f(u^{-1}g)\,du$$

for all $f \in \operatorname{ind}_{K_x}^{G(F)}(V_{\rho_x})$ and $g \in G(F)$, where du denotes the Haar measure on the pro-p-group $K_y \cap U(F)$ such that the volume of $K_y \cap U(F)$ is one.

Proof. Since the representation ρ_y is trivial on the group $K_y \cap U(F)$ and the restriction of ρ_y to $K_{y,+}$ is θ_y-isotypic, the character θ_y is trivial on the group $K_{y,+} \cap U(F) = K_y \cap U(F)$. Hence, by Lemma 3.4.7, we obtain that

$$\begin{aligned}\left(\Theta_{y|x}(f)\right)(g) &= |K_{y,+}/\left(K_{x,+}\cap K_{y,+}\right)|^{-1}\sum_{[k]\in K_{y,+}/(K_{x,+}\cap K_{y,+})}\theta_y(k)\cdot f(k^{-1}g)\\ &= |\left(K_y\cap U(F)\right)/\left(K_x\cap U(F)\right)|^{-1}\sum_{[u]\in(K_y\cap U(F))/(K_x\cap U(F))}\theta_y(u)\cdot f(u^{-1}g)\\ &= \int_{K_y\cap U(F)}\theta_y(u)\cdot f(u^{-1}g)\,du\\ &= \int_{K_y\cap U(F)} f(u^{-1}g)\,du.\end{aligned}$$

□

Corollary 3.5.4. *Let $x, y \in \mathcal{A}_{\text{gen}}$ and assume Axiom 3.4.1. For $v \in V_{\rho_x}$, we define $f_v \in \operatorname{ind}_{K_x}^{G(F)}(V_{\rho_x})$ as*

$$f_v(g) = \begin{cases}\rho_x(g)(v) & (g \in K_x),\\ 0 & (\textit{otherwise}).\end{cases}$$

Then we have

$$\left(\left(\Theta_{x|y}\circ\Theta_{y|x}\right)(f_v)\right)(1) = |K_y/\left(K_x\cap K_y\right)|^{-1}v,$$

where we regard $|K_y/\left(K_x\cap K_y\right)|$ as an element of $\mathcal{C}$. This element is invertible by Lemma 3.4.8. In particular, $\Theta_{y|x}$ is non-zero.

Proof. By Axiom 3.4.1(5), there exists $U \in \mathcal{U}(M)$ such that

$$K_x \cap U(F) \subseteq K_y \cap U(F) \quad \text{and} \quad K_y \cap \overline{U}(F) \subseteq K_x \cap \overline{U}(F).$$

Then according to Lemma 3.4.7 and Lemma 3.5.3, we have

$$\begin{aligned}
\left(\left(\Theta_{x|y} \circ \Theta_{y|x}\right)(f_v)\right)(1) &= \int_{K_x \cap \overline{U}(F)} \left(\Theta_{y|x}(f_v)\right)(\overline{u}^{-1})\, d\overline{u} \\
&= \int_{K_x \cap \overline{U}(F)} \int_{K_y \cap U(F)} f_v(u^{-1}\overline{u}^{-1})\, du\, d\overline{u} \\
&= \int_{K_x \cap \overline{U}(F)} \int_{K_x \cap K_y \cap U(F)} \rho_x(u^{-1}\overline{u}^{-1})(v)\, du\, d\overline{u} \\
&= \int_{K_x \cap \overline{U}(F)} \int_{K_x \cap U(F)} v\, du\, d\overline{u} \\
&= \left|(K_y \cap U(F)) \,/\, (K_x \cap U(F))\right|^{-1} v \\
&= \left|K_y / \left(K_x \cap K_y\right)\right|^{-1} v
\end{aligned}$$

□

We will prove a transitivity property of $\Theta_{y|x}$.

Proposition 3.5.5. *We assume Axiom 3.4.1. Let $x, y, z \in \mathcal{A}_{\mathrm{gen}}$ such that*

$$d(x, y) + d(y, z) = d(x, z).$$

Then we have

$$\Theta_{z|y} \circ \Theta_{y|x} = \Theta_{z|x}.$$

Proof. By using Axiom 3.4.1(5), there exists $U \in \mathcal{U}(M)$ such that

$$K_x \cap U(F) \subset K_y \cap U(F) \subset K_z \cap U(F)$$

and

$$K_z \cap \overline{U}(F) \subset K_y \cap \overline{U}(F) \subset K_x \cap \overline{U}(F).$$

Then according to Lemma 3.5.3, for $f \in \mathrm{ind}_{K_x}^{G(F)}(V_{\rho_x})$ and $g \in G(F)$, we have

$$\begin{aligned}
\left(\left(\Theta_{z|y} \circ \Theta_{y|x}\right)(f)\right)(g) &= \int_{K_z \cap U(F)} \left(\Theta_{y|x}(f)\right)(u_z^{-1}g)\, du_z \\
&= \int_{K_z \cap U(F)} \int_{K_y \cap U(F)} f(u_y^{-1}u_z^{-1}g)\, du_y\, du_z \\
&= \int_{K_y \cap U(F)} \int_{K_z \cap U(F)} f(u_y^{-1}u_z^{-1}g)\, du_z\, du_y \\
&= \int_{K_y \cap U(F)} \int_{K_z \cap U(F)} f(u_z^{-1}g)\, du_z\, du_y \\
&= \int_{K_z \cap U(F)} f(u_z^{-1}g)\, du_z \\
&= \left(\Theta_{z|x}(f)\right)(g).
\end{aligned}$$

□

We will prove a similar result under a weaker condition below (see Proposition 3.5.15). To do this, we define the notion of $\mathcal{K}$-relevance for an affine hyperplane $H \in \mathfrak{H}$. The group generated by the reflections across these $\mathcal{K}$-relevant hyperplanes will form (under additional assumptions) a basis for a subalgebra of $\mathcal{H}(G(F), \rho_{x_0})$ that is isomorphic to an affine Hecke algebra. (See Theorem 3.10.10.)

Definition 3.5.6. We say that an affine hyperplane $H \in \mathfrak{H}$ is $\mathcal{K}$-*relevant* if there exists $x, y \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$ and

$$\Theta_{x|y} \circ \Theta_{y|x} \notin \mathcal{C} \cdot \text{id}_{\text{ind}_{K_x}^{G(F)}(\rho_x)}. \tag{3.5.7}$$

We denote the set of $\mathcal{K}$-relevant hyperplanes in $\mathfrak{H}$ by $\mathfrak{H}_{\mathcal{K}\text{-rel}}$.

We will later see (in Corollary 3.5.20) that the action of $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ preserves the set $\mathfrak{H}_{\mathcal{K}\text{-rel}}$.

Remark 3.5.8. By the definition of $\mathfrak{H}_{\mathcal{K}\text{-rel}}$, an element $H \in \mathfrak{H}$ is contained in $\mathfrak{H}_{\mathcal{K}\text{-rel}}$ if and only if there exists $x, y \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$ and (3.5.7) holds. However, under additional assumptions, we will see in Proposition 3.8.18 below that if $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, then we have that (3.5.7) holds for *all* $x, y \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$.

The $\mathcal{K}$-relevant hyperplanes are those for which one has a more complicated description of $\Theta_{x|y} \circ \Theta_{y|x}$ for x and y on opposite sides of the hyperplane (see Lemma 3.8.9, Proposition 3.8.19). On the other hand, if $H \in \mathfrak{H}$ is not $\mathcal{K}$-relevant, then, according to the definition combined with Corollary 3.5.4, we have

$$\Theta_{x|y} \circ \Theta_{y|x} = |K_y / (K_x \cap K_y)|^{-1} \cdot \text{id}_{\text{ind}_{K_x}^{G(F)}(\rho_x)}$$

for all $x, y \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$.

We now normalize the operators $\Theta_{x|y}$ so that

$$\Theta^{\text{norm}}_{x|y} \circ \Theta^{\text{norm}}_{y|x} = \text{id}_{\text{ind}_{K_x}^{G(F)}(\rho_x)}$$

in this case (see Lemma 3.5.11). Recall that for an element $c = p^n \in \mathcal{C}$ with $n \in \mathbb{Z}$, we write $c^{1/2} := \left(p^{1/2}\right)^n$, where $p^{1/2}$ denotes the fixed square root of p in $\mathcal{C}$.

Definition 3.5.9. For $x, y \in \mathcal{A}_{\text{gen}}$, we define the $G(F)$-equivariant map

$$\Theta^{\text{norm}}_{y|x} \colon \text{ind}_{K_x}^{G(F)}(\rho_x) \longrightarrow \text{ind}_{K_y}^{G(F)}(\rho_y)$$

by

$$\Theta^{\text{norm}}_{y|x} = |K_y / (K_x \cap K_y)|^{1/2} \cdot \Theta_{y|x}.$$

Lemma 3.5.10. *Suppose that $H \in \mathfrak{H}$ is not $\mathcal{K}$-relevant. Then for all $x, y \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$, we have*

$$|K_x / (K_x \cap K_y)| = |K_y / (K_x \cap K_y)|.$$

Note that the image of this number in $\mathcal{C}$ is invertible by Lemma 3.4.8.

Proof. According to Corollary 3.5.4, we have

$$\Theta_{x|y} \circ \Theta_{y|x} = |K_y / (K_x \cap K_y)|^{-1} \cdot \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)}$$

and

$$\Theta_{y|x} \circ \Theta_{x|y} = |K_x / (K_x \cap K_y)|^{-1} \cdot \mathrm{id}_{\mathrm{ind}_{K_y}^{G(F)}(\rho_y)}.$$

Hence, we have

$$\begin{aligned}
\Theta_{y|x} &= \Theta_{y|x} \circ \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)} \\
&= |K_y / (K_x \cap K_y)| \, \Theta_{y|x} \circ \left(\Theta_{x|y} \circ \Theta_{y|x}\right) \\
&= |K_y / (K_x \cap K_y)| \left(\Theta_{y|x} \circ \Theta_{x|y}\right) \circ \Theta_{y|x} \\
&= |K_y / (K_x \cap K_y)| \, |K_x / (K_x \cap K_y)|^{-1} \cdot \mathrm{id}_{\mathrm{ind}_{K_y}^{G(F)}(\rho_y)} \circ \Theta_{y|x} \\
&= |K_y / (K_x \cap K_y)| \, |K_x / (K_x \cap K_y)|^{-1} \, \Theta_{y|x}.
\end{aligned}$$

Since $\Theta_{y|x}$ is non-zero, we obtain the lemma. □

Lemma 3.5.11. *Suppose that $H \in \mathfrak{H}$ is not $\mathcal{K}$-relevant. Then we have*

$$\Theta^{\mathrm{norm}}_{x|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)}$$

for all $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$.

Proof. According to Lemma 3.5.10, we have $|K_x / (K_x \cap K_y)| = |K_y / (K_x \cap K_y)|$. Then the claim follows from combining this with Corollary 3.5.4 and Definition 3.5.9. □

As above, we also obtain a transitivity property for the normalized intertwining operators $\Theta^{\mathrm{norm}}_{x|y}$.

Lemma 3.5.12. *Let $x, y, z \in \mathcal{A}_{\mathrm{gen}}$. Suppose that*

$$d(x, y) + d(y, z) = d(x, z).$$

Then we have

$$\Theta^{\mathrm{norm}}_{z|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{z|x}.$$

Proof. By using Axiom 3.4.1(5), we have $U \in \mathcal{U}(M)$ such that

$$K_x \cap U(F) \subset K_y \cap U(F) \subset K_z \cap U(F)$$

and

$$K_z \cap \overline{U}(F) \subset K_y \cap \overline{U}(F) \subset K_x \cap \overline{U}(F).$$

According to Lemma 3.4.7, we have

$$\begin{cases}
|K_y / (K_x \cap K_y)| &= |(K_y \cap U(F)) / (K_x \cap U(F))|, \\
|K_z / (K_y \cap K_z)| &= |(K_z \cap U(F)) / (K_y \cap U(F))|, \\
|K_z / (K_x \cap K_z)| &= |(K_z \cap U(F)) / (K_x \cap U(F))|.
\end{cases}$$

Thus, we obtain

$$\begin{aligned}|K_y/(K_x\cap K_y)|\,|K_z/(K_y\cap K_z)| &= |(K_y\cap U(F))\,/\,(K_x\cap U(F))|\,|(K_z\cap U(F))\,/\,(K_y\cap U(F))| \\ &= |(K_z\cap U(F))\,/\,(K_x\cap U(F))| \\ &= |K_z/(K_x\cap K_z)|\,.\end{aligned}$$

Combining it with Proposition 3.5.5, we obtain the claim. □

The transitivity property holds under a weaker assumption that only takes the $\mathcal{K}$-relevant hyperplanes into account. In order to state the weaker assumption and stronger result (see Proposition 3.5.15), we first need to introduce some notation.

Notation 3.5.13. For $x, y \in \mathcal{A}_{\mathrm{gen}}$, we define the subset $\mathfrak{H}_{\mathcal{K}\text{-rel};x,y}$ of $\mathfrak{H}_{\mathcal{K}\text{-rel}}$ by

$$\mathfrak{H}_{\mathcal{K}\text{-rel};x,y} = \{H \in \mathfrak{H}_{\mathcal{K}\text{-rel}} \mid x \text{ and } y \text{ are on opposite sides of } H\}\,,$$

and write $d_{\mathcal{K}\text{-rel}}(x,y) = \#\mathfrak{H}_{\mathcal{K}\text{-rel};x,y}$.

Lemma 3.5.14. *Let $x, y, z \in \mathcal{A}_{\mathrm{gen}}$. Then we have*

$$d_{\mathcal{K}\text{-rel}}(x,y) + d_{\mathcal{K}\text{-rel}}(y,z) \geq d_{\mathcal{K}\text{-rel}}(x,z),$$

and the following conditions are equivalent:

$$\begin{array}{ll}\text{(a)} & d_{\mathcal{K}\text{-rel}}(x,y) + d_{\mathcal{K}\text{-rel}}(y,z) = d_{\mathcal{K}\text{-rel}}(x,z), \\ \text{(b)} & \mathfrak{H}_{\mathcal{K}\text{-rel};x,y}, \mathfrak{H}_{\mathcal{K}\text{-rel};y,z} \subset \mathfrak{H}_{\mathcal{K}\text{-rel};x,z}, \\ \text{(c)} & \mathfrak{H}_{\mathcal{K}\text{-rel};x,y} \cap \mathfrak{H}_{\mathcal{K}\text{-rel};y,z} = \emptyset.\end{array}$$

Moreover, the condition

$$d(x,y) + d(y,z) = d(x,z)$$

implies

$$d_{\mathcal{K}\text{-rel}}(x,y) + d_{\mathcal{K}\text{-rel}}(y,z) = d_{\mathcal{K}\text{-rel}}(x,z).$$

Proof. The first and second claim follow from the same arguments as in the proof of Lemma 3.2.1. The last claim follows from the fact $\mathfrak{H}_{\mathcal{K}\text{-rel};x,y} = \mathfrak{H}_{\mathcal{K}\text{-rel}} \cap \mathfrak{H}_{x,y}$. □

Proposition 3.5.15. *Let $x, y, z \in \mathcal{A}_{\mathrm{gen}}$ and assume Axiom 3.4.1. Suppose that*

$$d_{\mathcal{K}\text{-rel}}(x,y) + d_{\mathcal{K}\text{-rel}}(y,z) = d_{\mathcal{K}\text{-rel}}(x,z).$$

Then we have

$$\Theta^{\mathrm{norm}}_{z|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{z|x}.$$

Proof. We use induction on $d(x,y)$. If $d(x,y) = 0$, we have

$$d(x,z) = d(y,z) = d(x,y) + d(y,z).$$

Hence, the proposition follows from Lemma 3.5.12. Suppose that $d(x,y) > 0$. We take $x' \in \mathcal{A}_{\mathrm{gen}}$ such that $d(x,x') = 1$ and $d(x,x') + d(x',y) = d(x,y)$. According to Lemma 3.5.12, we have

$$\Theta^{\mathrm{norm}}_{y|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{y|x}. \tag{3.5.15a}$$

Recall that we are assuming

$$d_{\mathcal{K}\text{-rel}}(x,y) + d_{\mathcal{K}\text{-rel}}(y,z) = d_{\mathcal{K}\text{-rel}}(x,z). \tag{3.5.15b}$$

On the other hand, since $d(x,x') + d(x',y) = d(x,y)$, according to Lemma 3.5.14, we also have

$$d_{\mathcal{K}\text{-rel}}(x,x') + d_{\mathcal{K}\text{-rel}}(x',y) = d_{\mathcal{K}\text{-rel}}(x,y). \tag{3.5.15c}$$

Combining (3.5.15b) with (3.5.15c), we obtain

$$\begin{aligned} d_{\mathcal{K}\text{-rel}}(x,z) &= d_{\mathcal{K}\text{-rel}}(x,y) + d_{\mathcal{K}\text{-rel}}(y,z) \\ &= d_{\mathcal{K}\text{-rel}}(x,x') + d_{\mathcal{K}\text{-rel}}(x',y) + d_{\mathcal{K}\text{-rel}}(y,z) \\ &\geq d_{\mathcal{K}\text{-rel}}(x,x') + d_{\mathcal{K}\text{-rel}}(x',z) \\ &\geq d_{\mathcal{K}\text{-rel}}(x,z). \end{aligned}$$

Since all of the inequalities above are thus equalities, we have

$$d_{\mathcal{K}\text{-rel}}(x',y) + d_{\mathcal{K}\text{-rel}}(y,z) = d_{\mathcal{K}\text{-rel}}(x',z) \tag{3.5.15d}$$

and

$$d_{\mathcal{K}\text{-rel}}(x,x') + d_{\mathcal{K}\text{-rel}}(x',z) = d_{\mathcal{K}\text{-rel}}(x,z). \tag{3.5.15e}$$

Then (3.5.15d) and the induction hypothesis imply that

$$\Theta^{\mathrm{norm}}_{z|y} \circ \Theta^{\mathrm{norm}}_{y|x'} = \Theta^{\mathrm{norm}}_{z|x'}. \tag{3.5.15f}$$

Combining (3.5.15a) with (3.5.15f), we have

$$\Theta^{\mathrm{norm}}_{z|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{z|y} \circ \Theta^{\mathrm{norm}}_{y|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{z|x'} \circ \Theta^{\mathrm{norm}}_{x'|x}.$$

Thus it suffices to show that $\Theta^{\mathrm{norm}}_{z|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{z|x}$. Let $H \subset \mathcal{A}_{x_0}$ denote the unique affine hyperplane in $\mathfrak{H}$ such that x and x' are on opposite sides of H. If x' and z are on the same side of H, then we have $\mathfrak{H}_{x,x'} \cap \mathfrak{H}_{x',z} = \emptyset$. Then, according to Lemma 3.2.1, we have $d(x,x') + d(x',z) = d(x,z)$. In this case, the claim follows from Lemma 3.5.12. Suppose that x' and z are on opposite sides of H. In this case, x and z are on the same side of H. Hence, we obtain $\mathfrak{H}_{x',x} \cap \mathfrak{H}_{x,z} = \emptyset$, equivalently, $d(x',x) + d(x,z) = d(x',z)$. Then, according to Lemma 3.5.12, we have

$$\Theta^{\mathrm{norm}}_{z|x} \circ \Theta^{\mathrm{norm}}_{x|x'} = \Theta^{\mathrm{norm}}_{z|x'}. \tag{3.5.15g}$$

We also note that $H \notin \mathfrak{H}_{\mathcal{K}\text{-rel}}$. Indeed, if $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, we have

$$H \in \mathfrak{H}_{\mathrm{rel};x,x'} \cap \mathfrak{H}_{\mathrm{rel};x',z} \neq \emptyset,$$

contradicting (3.5.15e). Hence, we obtain that H is not $\mathcal{K}$-relevant. Then, according to Lemma 3.5.11, we have

$$\Theta^{\mathrm{norm}}_{x|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)}. \tag{3.5.15h}$$

Combining (3.5.15g) with (3.5.15h), we obtain

$$\Theta^{\mathrm{norm}}_{z|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{z|x} \circ \Theta^{\mathrm{norm}}_{x|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{z|x}. \qquad \square$$

We record a corollary of Proposition 3.5.15, that is a generalization of Lemma 3.5.10 and will be used in Section 4.5 below.

Corollary 3.5.16. *Let $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $d_{\mathcal{K}\text{-rel}}(x,y) = 0$ and assume Axiom 3.4.1. Then we have*

$$|K_x / (K_x \cap K_y)| = |K_y / (K_x \cap K_y)| .$$

Proof. Since $d_{\mathcal{K}\text{-rel}}(x,y) + d_{\mathcal{K}\text{-rel}}(y,x) = d_{\mathcal{K}\text{-rel}}(x,x) = 0$, Proposition 3.5.15 implies that $\Theta^{\mathrm{norm}}_{x|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)}$. Combining this with Corollary 3.5.4 and Definition 3.5.9, we obtain that

$$|K_x / (K_x \cap K_y)|^{1/2} \, |K_y / (K_x \cap K_y)|^{1/2} \, |K_y / (K_x \cap K_y)|^{-1} = 1.$$

Thus, we obtain the claim. □

Now we construct the non-zero element $\varphi_{x,w} \in \mathcal{H}(G(F), \rho_x)_w$ for $x \in \mathcal{A}_{\mathrm{gen}}$ and $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$. In order to do so, we make the following choice.

Choice 3.5.17. We fix a family of non-zero elements

$$\mathcal{T} = \{T_n \in \mathrm{Hom}_{K_M}\left({}^n\rho_M, \rho_M\right)\}_{n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}}$$

that satisfies the following conditions:

(1) We have $T_1 = \mathrm{id}_{\rho_M}$.

(2) For all $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ and $k \in K_M \cap N(\rho_M)^{\heartsuit}_{[x_0]_M}$, we have $T_{nk} = T_n \circ \rho_M(k)$.

We will later refine our choice of $\mathcal{T}$. See Choices 3.10.3 and 3.11.5.

Let $x \in \mathcal{A}_{\mathrm{gen}}$ and $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$. According to Axiom 3.4.1(4a), the pair (K_{nx}, ρ_{nx}) is a quasi-G-cover of (K_M, ρ_M). On the other hand, according to Lemma 3.4.10, the pair $(K_{nx}, {}^n\rho_x)$ is a quasi-G-cover of $(K_M, {}^n\rho_M)$. Hence, according to Lemma 3.3.5, we have

$$T_n \in \mathrm{Hom}_{K_M}({}^n\rho_M, \rho_M) = \mathrm{Hom}_{K_{nx}}\left({}^n\rho_x, \rho_{nx}\right).$$

This allows us to define a second type of intertwining operator, whose composition with an appropriate $\Theta^{\mathrm{norm}}_{y|x}$ will allow us to define $\varphi_{x,w} \in \mathcal{H}(G(F), \rho_x)_w$ in Definition 3.5.23 below.

Definition 3.5.18. For $x \in \mathcal{A}_{\mathrm{gen}}$ and $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ we define the isomorphism

$$c_{x,n} \colon \mathrm{ind}_{K_x}^{G(F)}(\rho_x) \xrightarrow{\sim} \mathrm{ind}_{K_{nx}}^{G(F)}(\rho_{nx})$$

by

$$c_{x,n}(f) \colon g \longmapsto T_n(f(n^{-1}g))$$

for $f \in \mathrm{ind}_{K_x}^{G(F)}\left(V_{\rho_x}\right)$ and $g \in G(F)$.

Lemma 3.5.19. *Let $x, y \in \mathcal{A}_{\mathrm{gen}}$ and $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$. Then we have*

$$\Theta_{ny|nx} \circ c_{x,n} = c_{y,n} \circ \Theta_{y|x} \quad \textit{and} \quad \Theta^{\mathrm{norm}}_{ny|nx} \circ c_{x,n} = c_{y,n} \circ \Theta^{\mathrm{norm}}_{y|x}.$$

Proof. By the definitions of $\Theta_{ny|nx}$ and $c_{x,n}$ we have for $f \in \operatorname{ind}_{K_x}^{G(F)}(V_{\rho_x})$ and $g \in G(F)$ that

$$\left(\left(\Theta_{ny|nx} \circ c_{x,n}\right)(f)\right)(g) = \left|K_{ny,+}/\left(K_{nx,+} \cap K_{ny,+}\right)\right|^{-1} \sum_{[k] \in K_{ny,+}/(K_{nx,+} \cap K_{ny,+})} \theta_{ny}(k) \cdot T_n(f(n^{-1}k^{-1}g)).$$

On the other hand, by the definitions of $\Theta_{y|x}$ and $c_{y,n}$ and Axiom 3.4.1(3), we have

$$\begin{aligned}\left(\left(c_{y,n} \circ \Theta_{y|x}\right)(f)\right)(g) &= T_n\left(\left|K_{y,+}/\left(K_{x,+} \cap K_{y,+}\right)\right|^{-1} \sum_{[k] \in K_{y,+}/(K_{x,+} \cap K_{y,+})} \theta_y(k) \cdot f(k^{-1}n^{-1}g)\right) \\ &= \left|K_{ny,+}/\left(K_{nx,+} \cap K_{ny,+}\right)\right|^{-1} \sum_{[k] \in K_{ny,+}/(K_{nx,+} \cap K_{ny,+})} \theta_y(n^{-1}kn) \cdot T_n(f(n^{-1}k^{-1}g)).\end{aligned}$$

Now the first claim follows from Lemma 3.4.11, and the second claim follows from the first claim, Definition 3.5.9, and Axiom 3.4.1(3). □

Corollary 3.5.20. *Assume Axiom 3.4.1 and let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$. Then we have $n(H) \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$.*

Proof. According to Axiom 3.4.1(2), we have $n(H) \in \mathfrak{H}$. We will prove that $n(H)$ is $\mathcal{K}$-relevant. Since $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, there exists $x, y \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$ and

$$\Theta_{x|y} \circ \Theta_{y|x} \notin \mathcal{C} \cdot \mathrm{id}_{\operatorname{ind}_{K_x}^{G(F)}(\rho_x)}.$$

According to Lemma 3.5.19, we have

$$\Theta_{ny|nx} = c_{y,n} \circ \Theta_{y|x} \circ {c_{x,n}}^{-1} \quad \text{and} \quad \Theta_{nx|ny} = c_{x,n} \circ \Theta_{x|y} \circ {c_{y,n}}^{-1}.$$

Hence, we obtain that

$$\Theta_{nx|ny} \circ \Theta_{ny|nx} = \left(c_{x,n} \circ \Theta_{x|y} \circ {c_{y,n}}^{-1}\right) \circ \left(c_{y,n} \circ \Theta_{y|x} \circ {c_{x,n}}^{-1}\right) = c_{x,n} \circ \Theta_{x|y} \circ \Theta_{y|x} \circ {c_{x,n}}^{-1}.$$

Since

$$\Theta_{x|y} \circ \Theta_{y|x} \notin \mathcal{C} \cdot \mathrm{id}_{\operatorname{ind}_{K_x}^{G(F)}(\rho_x)},$$

we have

$$\Theta_{nx|ny} \circ \Theta_{ny|nx} \notin \mathcal{C} \cdot \mathrm{id}_{\operatorname{ind}_{K_{nx}}^{G(F)}(\rho_{nx})}.$$

Since $\mathfrak{H}_{nx,ny} = n(\mathfrak{H}_{x,y}) = \{n(H)\}$, we conclude that $n(H)$ is $\mathcal{K}$-relevant. □

Lemma 3.5.21. *Let $x \in \mathcal{A}_{\text{gen}}$ and $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$. Then the isomorphism $c_{x,n}$ only depends on the image of n in $W(\rho_M)^{\heartsuit}_{[x_0]_M}$.*

Proof. Let $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ and $k \in N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M$. We will prove that $c_{x,n} = c_{x,nk}$. Since $K_M \subset M(F)_{x_0}$ acts trivially on $\mathcal{A}_{x_0}$, we have $nkx = nx$. Let $f \in \operatorname{ind}_{K_x}^{G(F)}(V_{\rho_x})$ and $g \in G(F)$.

Then we have

$$\begin{aligned}(c_{x,nk}(f))(g) &= T_{nk}(f(k^{-1}n^{-1}g)) \\ &= (T_n \circ \rho_M(k)) \left(\rho_x(k^{-1}) f(n^{-1}g)\right) \\ &= (T_n \circ \rho_M(k)) \left(\rho_M(k^{-1}) f(n^{-1}g)\right) \\ &= T_n(f(n^{-1}g)) \\ &= (c_{x,n}(f))(g).\end{aligned}$$

Thus, we obtain the lemma. □

As a consequence of the lemma, we may write

$$c_{x,w} \coloneqq c_{x,n} \tag{3.5.22}$$

for $x \in \mathcal{A}_{\mathrm{gen}}$, $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$, and n any lift of w in $N(\rho_M)^{\heartsuit}_{[x_0]_M}$.

Definition 3.5.23. For $x \in \mathcal{A}_{\mathrm{gen}}$ and $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ we define the element

$$\Phi_{x,w} \in \mathrm{End}_{G(F)}\left(\mathrm{ind}_{K_x}^{G(F)}(\rho_x)\right)$$

by

$$\Phi_{x,w} = c_{w^{-1}x,w} \circ \Theta^{\mathrm{norm}}_{w^{-1}x|x}$$

and let $\varphi_{x,w}$ denote the element of $\mathcal{H}(G(F), \rho_x)$ that corresponds to $\Phi_{x,w}$ via the isomorphism in (2.2.3). We write $\varphi_w \coloneqq \varphi_{x_0,w}$ and $\Phi_w \coloneqq \Phi_{x_0,w}$.

Remark 3.5.24. Since $\Theta^{\mathrm{norm}}_{w^{-1}x|x}$ is non-zero and $c_{w^{-1}x,w}$ is an isomorphism, we obtain that $\Phi_{x,w}$ and $\varphi_{x,w}$ are non-zero. We will see below in Corollary 3.8.13 that under additional assumptions the endomorphism $\Phi_{x,w}$ is an isomorphism.

Remark 3.5.25. The elements $\Phi_{x,w}$ and $\varphi_{x,w}$ depend on the choice of the family

$$\mathcal{T} = \{T_n \in \mathrm{Hom}_{K_M}\left({}^n\rho_M, \rho_M\right)\}_{n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}}$$

made in Choice 3.5.17 as follows. Let $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ and $c \in \mathcal{C}^{\times}$. If we replace T_n with $c \cdot T_n$ for all lifts n of w in $N(\rho_M)^{\heartsuit}_{[x_0]_M}$, then the elements $\Phi_{x,w}$ and $\varphi_{x,w}$ are replaced by $c \cdot \Phi_{x,w}$ and $c \cdot \varphi_{x,w}$, respectively. We have chosen $T_1 = \mathrm{id}_{\rho_M}$ so that the endomorphism $\Phi_{x,1}$ is the identity map on $\mathrm{ind}_{K_x}^{G(F)}(\rho_x)$ for every $x \in \mathcal{A}_{\mathrm{gen}}$.

Lemma 3.5.26. *Let $x \in \mathcal{A}_{\mathrm{gen}}$ and $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$. Then we have* $\mathrm{supp}(\varphi_{x,w}) = K_x w K_x$. *Moreover, if n is a lift of w in $N(\rho_M)^{\heartsuit}_{[x_0]_M}$, then $(\varphi_{x,w}(n))v = |K_x/(K_{wx} \cap K_x)|^{-1/2}\, T_n(v)$ for all $v \in V_{\rho_x}$.*

Proof. We fix a lift n of w in $N(\rho_M)^{\heartsuit}_{[x0]_M}$. For $g \in G(F)$ and $v \in V_{\rho_x}$, we obtain using (2.2.4) and Axiom 3.4.1(3) that

$$\begin{aligned}(\varphi_{x,w}(g))v &= (\Phi_{x,w}(f_v))\,(g) \\ &= \left((c_{w^{-1}x,w} \circ \Theta^{\mathrm{norm}}_{w^{-1}x|x})(f_v)\right)(g) \\ &= T_n\left((\Theta^{\mathrm{norm}}_{w^{-1}x|x}(f_v))(n^{-1}g)\right) \\ &= |K_{w^{-1}x}/\left(K_x \cap K_{w^{-1}x}\right)|^{-1/2}\, T_n\left(\sum_{[k]\in K_{w^{-1}x,+}/\left(K_{x,+}\cap K_{w^{-1}x,+}\right)} \theta_{w^{-1}x}(k)\cdot f_v(k^{-1}n^{-1}g)\right) \\ &= |K_{w^{-1}x}/\left(K_x \cap K_{w^{-1}x}\right)|^{-1/2}\, T_n\left(\sum_{[k]\in K_{w^{-1}x,+}/\left(K_{x,+}\cap K_{w^{-1}x,+}\right)} \theta_{w^{-1}x}(k)\cdot f_v(n^{-1}(nkn^{-1})^{-1}g)\right) \\ &= |K_x/\left(K_{wx} \cap K_x\right)|^{-1/2}\, T_n\left(\sum_{[k]\in K_{x,+}/(K_{wx,+}\cap K_{x,+})} \theta_{w^{-1}x}(n^{-1}kn)\cdot f_v(n^{-1}k^{-1}g)\right).\end{aligned}$$

Since f_v is supported on K_x, the sum vanishes unless $g \in K_{x,+}nK_x \subseteq K_xwK_x$. Hence, since $\varphi_{x,w}$ is non-zero, we obtain that $\mathrm{supp}(\varphi_{x,w}) = K_xwK_x$. Moreover, we have

$$\sum_{[k]\in K_{x,+}/(K_{wx,+}\cap K_{x,+})} \theta_{w^{-1}x}(n^{-1}kn)\cdot f_v(n^{-1}k^{-1}n) = v,$$

which yields the second claim. □

Corollary 3.5.27. *Let $x \in \mathcal{A}_{x_0}$ and assume Axioms 3.4.1 and 3.4.3. As a vector space, we have*

$$\mathcal{H}(G(F), \rho_x) = \bigoplus_{w \in W(\rho_M)^{\heartsuit}_{[x0]_M}} \mathcal{C}\cdot\varphi_{x,w}.$$

Proof. The corollary follows from Proposition 3.4.18 and Lemma 3.5.26. □

3.6 Relations in the length-additive case

We will now study the structure of the Hecke algebra $\mathcal{H}(G(F), \rho_{x_0})$ as a $\mathcal{C}$-algebra. In this subsection, we investigate the relations in the length-additive case for a length that we define below in Definition 3.6.3. We keep the notation from the previous subsection and assume Axiom 3.4.1 (but not Axiom 3.4.3). Let $m, n \in N(\rho_M)^{\heartsuit}_{[x0]_M}$. Recall that we have fixed an isomorphism

$$T_n \in \mathrm{Hom}_{K_M}\left({}^{n}\rho_M, \rho_M\right) \subset \mathrm{End}_{\mathcal{C}}(V_{\rho_M})$$

in Choice 3.5.17. Since the subspaces $\mathrm{Hom}_{K_M}({}^{mn}\rho_M, {}^{m}\rho_M)$ and $\mathrm{Hom}_{K_M}({}^{n}\rho_M, \rho_M)$ of $\mathrm{End}_{\mathcal{C}}(V_{\rho_M})$ are equal, T_n is also an element of $\mathrm{Hom}_{K_M}({}^{mn}\rho_M, {}^{m}\rho_M)$. Then we can form the composition $T_m \circ T_n \in \mathrm{Hom}_{K_M}\left({}^{mn}\rho_M, \rho_M\right)$. Since $\dim_{\mathcal{C}}\left(\mathrm{Hom}_{K_M}\left({}^{mn}\rho_M, \rho_M\right)\right) = 1$, the isomorphisms $T_m \circ T_n$ and T_{mn} differ by a non-zero scalar, and it is straightforward to see that this scalar depends only on the images $\bar{m}$ and $\bar{n}$ of m and n in $W(\rho_M)^{\heartsuit}_{[x0]_M}$.

Notation 3.6.1. We denote by

$$\mu^{\mathcal{T}} : W(\rho_M)^{\heartsuit}_{[x_0]_M} \times W(\rho_M)^{\heartsuit}_{[x_0]_M} \to \mathcal{C}^{\times}$$

the unique map that satisfies

$$T_m \circ T_n = \mu^{\mathcal{T}}(\bar{m}, \bar{n}) \cdot T_{mn}.$$

Standard arguments of projective representations imply that $\mu^{\mathcal{T}}$ is a 2-cocycle. We note that the 2-cocycle $\mu^{\mathcal{T}}$ depends on the choice of a family $\mathcal{T} = \{T_n \in \mathrm{Hom}_{K_M}({}^{n}\rho_M, \rho_M)\}_{n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}}$, made in Choice 3.5.17, but its cohomology class does not. We will see in the next lemma that the 2-cocycle also determines the composition of the homomorphisms $c_{v^{-1}x,v}$, which we will then use to study the composition of the operators Φ_v.

Lemma 3.6.2. *Let $x \in \mathcal{A}_{\mathrm{gen}}$ and $v, w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$. Then the homomorphism*

$$c_{v^{-1}x,v} \circ c_{w^{-1}v^{-1}x,w} \in \mathrm{Hom}_{G(F)}\left(\mathrm{ind}_{K_{w^{-1}v^{-1}x}}^{G(F)}(\rho_{w^{-1}v^{-1}x}), \mathrm{ind}_{K_x}^{G(F)}(\rho_x)\right)$$

is equal to $\mu^{\mathcal{T}}(v, w) \cdot c_{w^{-1}v^{-1}x,vw}$.

Proof. We fix a lift m of v and n of w in $N(\rho_M)^{\heartsuit}_{[x_0]_M}$. For $f \in \mathrm{ind}_{K_{w^{-1}v^{-1}x}}^{G(F)}(\rho_{w^{-1}v^{-1}x})$ and $g \in G(F)$, we have

$$\begin{aligned}
\left(\left(c_{v^{-1}x,v} \circ c_{w^{-1}v^{-1}x,w}\right)(f)\right)(g) &= T_m\left(\left(c_{w^{-1}v^{-1}x,w}(f)\right)(m^{-1}g)\right) \\
&= T_m\left(T_n\left(f(n^{-1}m^{-1}g)\right)\right) \\
&= \left(T_m \circ T_n\right)\left(f(n^{-1}m^{-1}g)\right) \\
&= \mu^{\mathcal{T}}(m, n) \cdot T_{mn}\left(f(n^{-1}m^{-1}g)\right) \\
&= \left(\mu^{\mathcal{T}}(v, w) \cdot c_{w^{-1}v^{-1}x,vw}(f)\right)(g).
\end{aligned}$$

Thus, we obtain the lemma. □

We equip the group $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ with the following length function.

Definition 3.6.3. For $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$, we define

$$\ell_{\mathcal{K}\text{-rel}}(w) = d_{\mathcal{K}\text{-rel}}(x_0, w^{-1}x_0).$$

Proposition 3.6.4. *Let $v, w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ and assume Axiom 3.4.1. If*

$$\ell_{\mathcal{K}\text{-rel}}(vw) = \ell_{\mathcal{K}\text{-rel}}(v) + \ell_{\mathcal{K}\text{-rel}}(w),$$

then we have

$$\Phi_v \Phi_w = \mu^{\mathcal{T}}(v, w) \cdot \Phi_{vw}.$$

Proof. We have

$$\begin{aligned} d_{\mathcal{K}\text{-rel}}(x_0, w^{-1}x_0) + d_{\mathcal{K}\text{-rel}}(w^{-1}x_0, w^{-1}v^{-1}x_0) &= d_{\mathcal{K}\text{-rel}}(x_0, w^{-1}x_0) + d_{\mathcal{K}\text{-rel}}(x_0, v^{-1}x_0) \\ &= d_{\mathcal{K}\text{-rel}}(x_0, v^{-1}x_0) + d_{\mathcal{K}\text{-rel}}(x_0, w^{-1}x_0) \\ &= \ell_{\mathcal{K}\text{-rel}}(v) + \ell_{\mathcal{K}\text{-rel}}(w) \\ &= \ell_{\mathcal{K}\text{-rel}}(vw) \\ &= d_{\mathcal{K}\text{-rel}}(x_0, w^{-1}v^{-1}x_0). \end{aligned}$$

Hence, according to Proposition 3.5.15, we have

$$\Theta^{\text{norm}}_{w^{-1}v^{-1}x_0|w^{-1}x_0} \circ \Theta^{\text{norm}}_{w^{-1}x_0|x_0} = \Theta^{\text{norm}}_{w^{-1}v^{-1}x_0|x_0}.$$

On the other hand, according to Lemma 3.5.19, we have

$$\Theta^{\text{norm}}_{v^{-1}x_0|x_0} \circ c_{w^{-1}x_0,w} = c_{w^{-1}v^{-1}x_0,w} \circ \Theta^{\text{norm}}_{w^{-1}v^{-1}x_0|w^{-1}x_0}.$$

Thus we obtain using Lemma 3.6.2

$$\begin{aligned} \Phi_v\Phi_w &= \left(c_{v^{-1}x_0,v} \circ \Theta^{\text{norm}}_{v^{-1}x_0|x_0}.\right) \circ \left(c_{w^{-1}x_0,w} \circ \Theta^{\text{norm}}_{w^{-1}x_0|x_0}\right) \\ &= c_{v^{-1}x_0,v} \circ \left(\Theta^{\text{norm}}_{v^{-1}x_0|x_0} \circ c_{w^{-1}x_0,w}\right) \circ \Theta^{\text{norm}}_{w^{-1}x_0|x_0} \\ &= c_{v^{-1}x_0,v} \circ \left(c_{w^{-1}v^{-1}x_0,w} \circ \Theta^{\text{norm}}_{w^{-1}v^{-1}x_0|w^{-1}x_0}\right) \circ \Theta^{\text{norm}}_{w^{-1}x_0|x_0} \\ &= \left(c_{v^{-1}x_0,v} \circ c_{w^{-1}v^{-1}x_0,w}\right) \circ \left(\Theta^{\text{norm}}_{w^{-1}v^{-1}x_0|w^{-1}x_0} \circ \Theta^{\text{norm}}_{w^{-1}x_0|x_0}\right) \\ &= \left(\mu^{\mathcal{T}}(v,w) \cdot c_{w^{-1}v^{-1}x_0,vw}\right) \circ \Theta^{\text{norm}}_{w^{-1}v^{-1}x_0|x_0} \\ &= \mu^{\mathcal{T}}(v,w) \cdot \Phi_{vw}. \end{aligned}$$

□

3.7 The structure of the indexing group

We keep the notation from the previous subsection and assume Axioms 3.4.1 and 3.4.3. In this subsection, we will introduce an additional axiom about $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ containing a nice subgroup, Axiom 3.7.1, that allows us to deduce that $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ is a semi-direct product of a normal affine Weyl group with the subgroup of $\ell_{\mathcal{K}\text{-rel}}$-length-zero elements. This generalizes the decomposition that Morris ([Mor93, 7.3. Proposition]) obtains for his group $W(\sigma)$ (using his notation) that indexes a basis for the Hecke algebra of a depth-zero type attached to a parahoric subgroup.

For $H \in \mathfrak{H}$, let s_H denote the orthogonal reflection on $\mathcal{A}_{x_0}$ with respect to the affine hyperplane H. We define $W_{\mathcal{K}\text{-rel}}$ to be the subgroup of the affine transformations of $\mathcal{A}_{x_0}$ generated by $\{s_H \mid H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}\}$, i.e.,

$$W_{\mathcal{K}\text{-rel}} = \langle s_H \mid H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}\rangle.$$

Axiom 3.7.1. There exists a normal subgroup $W(\rho_M)_{\text{aff}}$ of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ such that the action of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ restricts to an isomorphism

$$W(\rho_M)_{\text{aff}} \xrightarrow{\sim} W_{\mathcal{K}\text{-rel}}.$$

One way to check that the axiom is satisfied is via the following lemma.

Lemma 3.7.2. *Assume Axioms 3.4.1 and 3.4.3, and suppose that there exists a normal subgroup G' of $G(F)$ such that*

$$G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M = G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap M(F)_{x_0}$$

and that for all $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, there exists an element

$$s'_H \in \left(G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M}\right) / \left(G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right)$$

such that the action of s'_H on $\mathcal{A}_{x_0}$ agrees with the orthogonal reflection s_H. Then Axiom 3.7.1 is satisfied with

$$W(\rho_M)_{\text{aff}} = \langle s'_H \mid H \in \mathfrak{H}_{\mathcal{K}\text{-rel}} \rangle.$$

In the depth-zero setting, i.e., in Section 5, we will apply this lemma in the case that the normal subgroup G' is the kernel of the Kottwitz homomorphism to prove that Axiom 3.7.1 is satisfied, see Proposition 5.3.5. In the setting of [FOAM], we will observe that $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ there is the same as the corresponding group in a depth-zero setting and the relevant hyperplanes are a subset of the hyperplanes in the depth-zero setting, so that Axiom 3.7.1 in the setting of [FOAM] follows from the result in the depth-zero setting, see [FOAM, Proposition 4.3.9].

Proof of Lemma 3.7.2. By Remark 3.4.16 and Corollary 3.5.20, the group

$$\left(G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M}\right) / \left(G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right)$$

acts faithfully on $\mathcal{A}_{x_0}$ through affine transformations and preserves $\mathfrak{H}_{\mathcal{K}\text{-rel}}$. Thus we can identify the former group with a subgroup of the group of affine transformations of $\mathcal{A}_{x_0}$ preserving $\mathfrak{H}_{\mathcal{K}\text{-rel}}$. Since the latter contains the group $W_{\mathcal{K}\text{-rel}} = \langle s_H \mid H \in \mathfrak{H}_{\mathcal{K}\text{-rel}} \rangle$ as a normal subgroup, the claim follows from the assumption about the existence of $s'_H \in \left(G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M}\right) / \left(G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right)$ and the observation that $\left(G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M}\right) / \left(G' \cap N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right)$ is a normal subgroup of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$. □

We will now show that the group $W(\rho_M)_{\text{aff}}$ of Axiom 3.7.1 is an affine Weyl group, which explains our choice of notation. For $H \in \mathfrak{H}$, let a_H denote an affine functional on $\mathcal{A}_{x_0}$ such that $H = \{x \in \mathcal{A}_{x_0} \mid a_H(x) = 0\}$. We write Da_H for the gradient of a_H, which is a linear functional on $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. The subspace

$$\ker(Da_H) = \{v \in X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} \mid Da_H(v) = 0\}$$

of $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$ only depends on H and not on the choice of a_H. We define the subspace $V^{\mathcal{K}\text{-rel}}$ of $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$V^{\mathcal{K}\text{-rel}} = \bigcap_{H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}} \ker(Da_H).$$

We define the affine space $\mathcal{A}_{\mathcal{K}\text{-rel}}$ to be the quotient affine space $\mathcal{A}_{x_0}/V^{\mathcal{K}\text{-rel}}$ and write its vector space of translations as

$$V_{\mathcal{K}\text{-rel}} = \left(X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}\right)/V^{\mathcal{K}\text{-rel}}.$$

According to Corollary 3.5.20, the action of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ induces a well-defined action of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{\mathcal{K}\text{-rel}}$. Let $(V^{\mathcal{K}\text{-rel}})^{\perp}$ denote the orthogonal complement of $V^{\mathcal{K}\text{-rel}}$ in $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$ with respect to the previously fixed $N_G(M)(F)$-invariant inner product $(\ ,\ )_M$ on $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. Then the natural projection $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} \to V_{\mathcal{K}\text{-rel}}$ restricts to an isomorphism

$$(V^{\mathcal{K}\text{-rel}})^{\perp} \xrightarrow{\sim} V_{\mathcal{K}\text{-rel}}. \tag{3.7.3}$$

We define an inner product on $V_{\mathcal{K}\text{-rel}}$ by restricting the inner product $(\ ,\ )_M$ to $(V^{\mathcal{K}\text{-rel}})^{\perp}$ and then transporting it to $V_{\mathcal{K}\text{-rel}}$ via the isomorphism in (3.7.3). This turns the affine space $\mathcal{A}_{\mathcal{K}\text{-rel}}$ into a Euclidean space. We identify an affine hyperplane $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ with its image on $\mathcal{A}_{\mathcal{K}\text{-rel}}$.

Proposition 3.7.4. *Assume Axioms 3.4.1, 3.4.3 and 3.7.1. Then there exists an affine root system $\Gamma(\rho_M)$ on $\mathcal{A}_{\mathcal{K}\text{-rel}}$ whose vanishing hyperplanes are $\mathfrak{H}_{\mathcal{K}\text{-rel}}$. In particular, the action of $W(\rho_M)_{\mathrm{aff}}$ on $\mathcal{A}_{\mathcal{K}\text{-rel}}$ induces an isomorphism*

$$W(\rho_M)_{\mathrm{aff}} \simeq W_{\mathcal{K}\text{-rel}} = W_{\mathrm{aff}}(\Gamma(\rho_M)),$$

where $W_{\mathrm{aff}}(\Gamma(\rho_M))$ denotes the affine Weyl group of $\Gamma(\rho_M)$. (We allow the affine root system to be empty, with the associated affine Weyl group being the trivial group.)

Proof. We assume that $\mathcal{A}_{\mathcal{K}\text{-rel}}$ has dimension at least one because the statement is otherwise trivial. According to [Bou68, Chapter V, Section 3.10, Proposition 10] and the proof of [Bou68, Chapter VI, Section 2.5, Proposition 8], it suffices to check the following conditions (see also the proof of [Mor93, 2.7 Theorem (b)]):

(1) For any $w \in W_{\mathcal{K}\text{-rel}}$ and $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, we have $w(H) \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$.

(2) The group $W_{\mathcal{K}\text{-rel}}$ acts properly on $\mathcal{A}_{\mathcal{K}\text{-rel}}$.

(3) For any $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, there are infinitely many $H' \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ that are parallel to H.

Condition (1) follows from Corollary 3.5.20 and Axiom 3.7.1, and Condition (2) follows from Lemma 3.4.17 and Axiom 3.7.1. It remains to prove Condition (3). Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$. We fix an affine functional a on $\mathcal{A}_{x_0}$ such that $H = \{x \in \mathcal{A}_{x_0} \mid a(x) = 0\}$. We write α for the gradient of a. Let $t \in A_M(F)$. Since $t \in A_M(F) \subset N(\rho_M)^{\heartsuit}_{[x_0]_M}$, we have $t(H) \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ by Corollary 3.5.20. We define the element $\nu(t) \in X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$ by $\chi(\nu(t)) = -\operatorname{ord}(\chi(t))$ for all $\chi \in X^*(A_M)$. We note that the set $\{\nu(t) \mid t \in A_M(F)\}$ is a lattice of full rank in $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} \neq \{0\}$. Hence, we can take $t \in A_M(F)$ such that $\alpha(\nu(t)) \neq 0$. According to [KP23, Proposition 6.2.4], we have $t^{-1}x = x - \nu(t)$. Hence, we obtain that

$$\begin{aligned} t(H) &= \{tx \in \mathcal{A}_{x_0} \mid a(x) = 0\} = \left\{x \in \mathcal{A}_{x_0} \mid a(t^{-1}x) = 0\right\} = \{x \in \mathcal{A}_{x_0} \mid a(x - \nu(t)) = 0\} \\ &= \{x \in \mathcal{A}_{x_0} \mid a(x) - \alpha(\nu(t)) = 0\}. \end{aligned}$$

Thus, we obtain that $t(H)$ is parallel to H. Moreover, since $\alpha(\nu(t)) \neq 0$, by applying the same calculations to t^n instead of t for $n \in \mathbb{Z}$, we obtain that the affine hyperplanes $t^n(H)$ with $n \in \mathbb{Z}$ are all pairwise distinct, contained in $\mathfrak{H}_{\mathcal{K}\text{-rel}}$, and parallel to H. □

We now construct a complement to $W(\rho_M)_{\mathrm{aff}}$ in $W(\rho_M)^{\heartsuit}_{[x_0]_M}$.

Notation 3.7.5. We call the connected components of the complement of $\mathfrak{H}_{\mathcal{K}\text{-rel}}$ in $\mathcal{A}_{\mathcal{K}\text{-rel}}$ *chambers* and denote by $C_{\mathcal{K}\text{-rel}}$ the chamber of $\mathcal{A}_{\mathcal{K}\text{-rel}}$ that contains $x_0+V^{\mathcal{K}\text{-rel}}$. We define the subgroup $\Omega(\rho_M)$ of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ by

$$\Omega(\rho_M) = \left\{ w \in W(\rho_M)^{\heartsuit}_{[x_0]_M} \,\middle|\, w(C_{\mathcal{K}\text{-rel}}) = C_{\mathcal{K}\text{-rel}} \right\}.$$

Note that $\Omega(\rho_M)$ consists precisely of the length-zero elements of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ with respect to the length $\ell_{\mathcal{K}\text{-rel}}$ introduced in Definition 3.6.3.

Proposition 3.7.6. *Assume Axioms 3.4.1, 3.4.3 and 3.7.1. Then we have*

$$W(\rho_M)^{\heartsuit}_{[x_0]_M} = \Omega(\rho_M) \ltimes W(\rho_M)_{\text{aff}}.$$

Proof. According to Proposition 3.7.4, the action of $W(\rho_M)_{\text{aff}}$ on the set of all chambers of $\mathcal{A}_{\mathcal{K}\text{-rel}}$ is simply transitive. Hence we obtain that

$$\Omega(\rho_M) \cap W(\rho_M)_{\text{aff}} = \{1\}$$

and

$$W(\rho_M)^{\heartsuit}_{[x_0]_M} = \Omega(\rho_M) \cdot W(\rho_M)_{\text{aff}}.$$

Since $W(\rho_M)_{\text{aff}}$ is a normal subgroup of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$, the proposition follows. □

3.8 Simple reflections and quadratic relations

In this subsection, we continue our study of the structure of the Hecke algebra $\mathcal{H}(G(F), \rho_{x_0})$. In Section 3.6, we investigated the relations of basis elements of the Hecke algebra in the length-additive case, see Proposition 3.6.4. In this subsection, we will study the relations in the basic non-length-additive case, the case of (simple) reflections, introduced in Notation 3.8.1 below. This will require one further axiom, Axiom 3.8.2 below, which ensures that the Hecke algebra element Φ_s corresponding to a simple reflection s satisfies a quadratic relation. This additional axiom also allows us to prove that the endomorphism Φ_w for $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ is invertible, see Corollary 3.8.13.

We keep the notation from the previous subsections and assume all the above axioms, i.e., Axioms 3.4.1, 3.4.3, and 3.7.1.

Notation 3.8.1. We denote by $S_{\mathcal{K}\text{-rel}} \subset W_{\mathcal{K}\text{-rel}}$ the subset of *simple reflections* corresponding to the chamber $C_{\mathcal{K}\text{-rel}}$, i.e., the reflections across the walls of $C_{\mathcal{K}\text{-rel}}$. Using the isomorphism of $W_{\mathcal{K}\text{-rel}}$ with the subgroup $W(\rho_M)_{\text{aff}}$ of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ from Axiom 3.7.1, we also view $S_{\mathcal{K}\text{-rel}}$ as a subset of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$. For each $s \in S_{\mathcal{K}\text{-rel}}$, we denote by $H_s \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ the corresponding wall of $C_{\mathcal{K}\text{-rel}}$.

We note that the restriction of the length function $\ell_{\mathcal{K}\text{-rel}}$ to $W(\rho_M)_{\text{aff}}$ agrees with the length function of the Coxeter system $(W(\rho_M)_{\text{aff}}, S_{\mathcal{K}\text{-rel}})$. We will now impose one more axiom that guarantees that the Hecke algebra elements supported on double cosets of lifts of simple reflections satisfy a quadratic relation.

Axiom 3.8.2. For any $s \in S_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,sx} = \{H_s\}$, there exists a compact, open subgroup $K'_{x,s}$ of $G(F)$ containing K_x such that

$$W(\rho_M)^{\heartsuit}_{[x_0]_M} \supset \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K'_{x,s}\right) / \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right) = \{1, s\}.$$

Remark 3.8.3. In the depth-zero setting discussed in Section 5, the axiom is satisfied for the group $K'_{x,s} = K_M \cdot G(F)_{h,0}$ for $h \in H_s$ the unique point for which $h = x + t \cdot (sx - x)$ for some $0 < t < 1$. For the more general setup in [FOAM] the axiom is satisfied for $K'_{x,s} = K_h$ as defined in [FOAM, Equation (4.1.3)]. This is proven in Corollary 4.3.10 of [FOAM].

Notation 3.8.4. For $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\text{gen}}$ with $\mathfrak{H}_{x,s_H x} = \{H\}$, let $K_{x,s_H} \coloneqq \langle K_x, s_H\rangle$. Here, we regard $s_H \in W_{\mathcal{K}\text{-rel}}$ as an element of $W(\rho_M)_{\text{aff}} \subset W(\rho_M)^{\heartsuit}_{[x_0]_M}$ via the isomorphism in Axiom 3.7.1.

Remark 3.8.5. If a group $K'_{x,s}$ exists as in Axiom 3.8.2, then we obtain that the axiom also holds for the group $K_{x,s}$ in place of $K'_{x,s}$. This is because $K_{x,s}$ is an open (hence closed) subgroup of $K'_{x,s}$, and thus is compact and

$$\begin{aligned}\{1, s\} &\subseteq \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_{x,s}\right) / \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right) \\ &\subseteq \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K'_{x,s}\right) / \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right) = \{1, s\}.\end{aligned}$$

Thus, we conclude that $\left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_{x,s}\right) / \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right) = \{1, s\}$.

The reason why we state Axiom 3.8.2 with an arbitrary subgroup $K'_{x,s}$ rather than with $K_{x,s}$ is that in Section 4 below, in order to obtain an isomorphism between different Hecke algebras, we will state that Axiom 3.8.2 is satisfied for a specific choice of $K'_{x,s}$, see the beginning of Section 4.4 and Theorem 4.4.8.

Remark 3.8.6. While Axiom 3.8.2 only concerns simple reflections, it implies the same result for other reflections. More precisely, let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,s_H x} = \{H\}$. According to [Bou68, Chapter V, Section 3.1, Lemma 2], there exists $w \in W(\rho_M)_{\text{aff}}$ and $s \in S_{\mathcal{K}\text{-rel}}$ such that $w(H) = H_s$ and hence $s_H = w^{-1} s w$. Thus, from Axiom 3.8.2 and Remark 3.8.5, we obtain that the group $K_{x,s_H} = \langle K_x, s_H\rangle = w^{-1} K_{wx,s} w$ is a compact, open subgroup of $G(F)$ and

$$\left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_{x,s_H}\right) / \left(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M\right) = \{1, s_H\}.$$

We remind the reader that we have shown in Section 2.2, in particular around Diagram (2.2.7), how to view $\text{End}_{K_{x,s_H}}\big(\text{ind}_{K_x}^{K_{x,s_H}}(\rho_x)\big)$ as a subalgebra of $\text{End}_{G(F)}\big(\text{ind}_{K_x}^{G(F)}(\rho_x)\big)$.

Lemma 3.8.7. *Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,s_H x} = \{H\}$. Then the elements Φ_{x,s_H} and $\Phi_{x,1}$ form a basis of $\text{End}_{K_{x,s_H}}\left(\text{ind}_{K_x}^{K_{x,s_H}}(\rho_x)\right)$.*

In particular, we have

$$\dim_{\mathcal{C}}\left(\text{End}_{K_{x,s_H}}\left(\text{ind}_{K_x}^{K_{x,s_H}}(\rho_x)\right)\right) = 2.$$

Proof. According to the isomorphism in (2.2.6), it suffices to show that the elements φ_{x,s_H} and $\varphi_{x,1}$ give a basis of $\mathcal{H}(K_{x,s_H},\rho_x)$. According to Proposition 3.4.18 and Lemma 3.5.26, a basis of the space $\mathcal{H}(K_{x,s_H},\rho_x)$ is given by the set

$$\left\{\varphi_{x,w}\,\middle|\, w\in\left(N(\rho_M)^{\heartsuit}_{[x_0]_M}\cap K_{x,s_H}\right)/\left(N(\rho_M)^{\heartsuit}_{[x_0]_M}\cap K_M\right)\right\}.$$

Thus, the lemma follows from Remark 3.8.6. □

Corollary 3.8.8. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. Let $H\in\mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x\in\mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,s_Hx}=\{H\}$. Then there exist $p_{x,H},q_{x,H}\in\mathcal{C}$ such that*

$$(\Phi_{x,s_H})^2=p_{x,H}\cdot\Phi_{x,s_H}+q_{x,H}\cdot\Phi_{x,1}.$$

Proof. Since Φ_{x,s_H} belongs to the subalgebra $\mathrm{End}_{K_{x,s_H}}\big(\mathrm{ind}_{K_x}^{K_{x,s_H}}(\rho_x)\big)$ of $\mathrm{End}_{G(F)}\big(\mathrm{ind}_{K_x}^{G(F)}(\rho_x)\big)$, the same is true of $(\Phi_{x,s_H})^2$. Thus, the corollary follows from Lemma 3.8.7. □

The remainder of this subsection is concerned with strengthening the statement of this Corollary by replacing the condition $\mathfrak{H}_{x,s_Hx}=\{H\}$ by the weaker condition $\mathfrak{H}_{\mathcal{K}\text{-rel};x,s_Hx}=\{H\}$ and by proving that the coefficients $p_{x,H}$ and $q_{x,H}$ in the above quadratic relation are non-zero and independent of the choice x. While $q_{x,H}\neq 0$ follows from a standard observation about the endomorphism $\Theta^{\mathrm{norm}}_{x|s_Hx}\circ\Theta^{\mathrm{norm}}_{s_Hx|x}$ for a general $H\in\mathfrak{H}$ and $x\in\mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,s_Hx}=\{H\}$, to prove $p_{x,H}\neq 0$, the condition $H\in\mathfrak{H}_{\mathcal{K}\text{-rel}}$ is essential. We remind the reader that the condition $H\in\mathfrak{H}_{\mathcal{K}\text{-rel}}$ was defined by the existence of $x,y\in\mathcal{A}_{\mathrm{gen}}$ with $\mathfrak{H}_{x,y}=\{H\}$ and $\Theta_{x|y}\circ\Theta_{y|x}$ a non-scalar operator on $\mathrm{ind}_{K_x}^{G(F)}(\rho_x)$. In order to prove $p_{x,H}\neq 0$, we will show that if $\Theta_{x|y}\circ\Theta_{y|x}$ is a non-scalar operator for some $x,y\in\mathcal{A}_{\mathrm{gen}}$ with $\mathfrak{H}_{x,y}=\{H\}$, then the same is true for *all* such $x,y\in\mathcal{A}_{\mathrm{gen}}$, and even for all $x,y\in\mathcal{A}_{\mathrm{gen}}$ satisfying only the weaker condition $\mathfrak{H}_{\mathcal{K}\text{-rel};x,y}=\{H\}$, see Proposition 3.8.18. Once we know that the coefficients $p_{x,H}$ and $q_{x,H}$ are non-zero and independent of x, we will refine our choice of $\mathcal{T}=\{T_n\in\mathrm{Hom}_{K_M}({}^n\rho_M,\rho_M)\}_{n\in N(\rho_M)^{\heartsuit}_{[x_0]_M}}$ that entered the definition of the endomorphism Φ_s to obtain a quadratic relation $(\Phi_s)^2=(q_s-1)\cdot\Phi_s+q_s\cdot\Phi_1$ with $q_s\in\mathcal{C}^\times\smallsetminus\{1\}$ for every simple reflection s, see Proposition 3.8.23.

We begin by proving that $q_{x,H}$ is non-zero and independent of x, for which we first record a consequence of Lemma 3.5.19 and Lemma 3.6.2:

Lemma 3.8.9. *Let $H\in\mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x\in\mathcal{A}_{\mathrm{gen}}$. Then we have*

$$(\Phi_{x,s_H})^2=\mu^{\mathcal{T}}(s_H,s_H)\cdot\Theta^{\mathrm{norm}}_{x|s_Hx}\circ\Theta^{\mathrm{norm}}_{s_Hx|x}.$$

Proof. We write $s=s_H$. Since $s^2=1$, Lemma 3.5.19 and Lemma 3.6.2 imply that

$$\begin{aligned}(\Phi_{x,s})^2&=c_{s^{-1}x,s}\circ\Theta^{\mathrm{norm}}_{s^{-1}x|x}\circ c_{s^{-1}x,s}\circ\Theta^{\mathrm{norm}}_{s^{-1}x|x}\\&=c_{s^{-1}x,s}\circ c_{x,s}\circ\Theta^{\mathrm{norm}}_{x|s^{-1}x}\circ\Theta^{\mathrm{norm}}_{s^{-1}x|x}\\&=\mu^{\mathcal{T}}(s,s)\cdot c_{x,1}\circ\Theta^{\mathrm{norm}}_{x|s^{-1}x}\circ\Theta^{\mathrm{norm}}_{s^{-1}x|x}\\&=\mu^{\mathcal{T}}(s,s)\cdot\Theta^{\mathrm{norm}}_{x|s^{-1}x}\circ\Theta^{\mathrm{norm}}_{s^{-1}x|x}\\&=\mu^{\mathcal{T}}(s,s)\cdot\Theta^{\mathrm{norm}}_{x|sx}\circ\Theta^{\mathrm{norm}}_{sx|x}.\end{aligned}$$

□

Lemma 3.8.10. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,s_H x} = \{H\}$, and let $q_{x,H}$ be as in Corollary 3.8.8. Then $q_{x,H} = \mu^{\mathcal{T}}(s_H, s_H)$. In particular, the scalar $q_{x,H}$ is invertible and independent of x.*

Proof. We write $s = s_H$. For $v \in V_{\rho_x}$, we define the element $f_v \in \mathrm{ind}_{K_x}^{G(F)}(V_{\rho_x})$ by

$$f_v(g) = \begin{cases} \rho_x(g)(v) & (g \in K_x), \\ 0 & (\text{otherwise}). \end{cases}$$

Then, according to Corollary 3.5.4, we have

$$\big((\Theta_{x|sx} \circ \Theta_{sx|x})(f_v)\big)(1) = |K_{sx}/(K_x \cap K_{sx})|^{-1} v.$$

Hence, we obtain from Definition 3.5.9 and Axiom 3.4.1(3) that

$$\big((\Theta^{\mathrm{norm}}_{x|sx} \circ \Theta^{\mathrm{norm}}_{sx|x})(f_v)\big)(1) = |K_x/(K_x \cap K_{sx})|^{1/2} |K_{sx}/(K_x \cap K_{sx})|^{-1/2} v = v.$$

Combining this with Lemma 3.8.9, we obtain that

$$\left(\left(\Phi_{x,s}^2\right)(f_v)\right)(1) = \mu^{\mathcal{T}}(s,s) \cdot v. \tag{3.8.10a}$$

Substituting $(\Phi_{x,s})^2$ by $p_{x,H} \cdot \Phi_{x,s} + q_{x,H} \cdot \Phi_{x,1}$ in (3.8.10a), we obtain $q_{x,H} \cdot v = \mu^{\mathcal{T}}(s,s) \cdot v$. Thus, $q_{x,H} = \mu^{\mathcal{T}}(s,s) \in \mathcal{C}^{\times}$, as desired. □

Corollary 3.8.11. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,s_H x} = \{H\}$. Then the endomorphism Φ_{x,s_H} is invertible.*

Proof. Recall from Remark 3.5.25 that $\Phi_{x,1}$ is the identity endomorphism. Hence by Corollary 3.8.8 and Lemma 3.8.10, the element $q_{x,H}^{-1} \cdot (\Phi_{x,s_H} - p_{x,H} \cdot \Phi_{x,1})$ is a left and right inverse of Φ_{x,s_H}. □

From this we can deduce that more generally all endomorphism $\Phi_{x,w}$ with $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ are invertible as well as the operators $\Theta_{x|y}$ for all $x, y \in \mathcal{A}_{\mathrm{gen}}$ that we used to define $\Phi_{x,w}$.

Proposition 3.8.12. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. Let $x, y \in \mathcal{A}_{\mathrm{gen}}$. Then the operators $\Theta_{y|x}$ and $\Theta^{\mathrm{norm}}_{y|x}$ are isomorphisms.*

Proof. It suffices to show that $\Theta^{\mathrm{norm}}_{y|x}$ is an isomorphism. If $d(x,y) = 0$, according to Lemma 3.5.12, we have

$$\Theta^{\mathrm{norm}}_{x|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{x|x} = \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)} \quad \text{and} \quad \Theta^{\mathrm{norm}}_{y|x} \circ \Theta^{\mathrm{norm}}_{x|y} = \Theta^{\mathrm{norm}}_{y|y} = \mathrm{id}_{\mathrm{ind}_{K_y}^{G(F)}(\rho_y)}.$$

Hence $\Theta^{\mathrm{norm}}_{y|x}$ is an isomorphism. Suppose that $d(x,y) \geq 1$. According to Lemma 3.5.12, we can take points

$$x_1 = x, x_2, x_3, \ldots, x_t = y \in \mathcal{A}_{\mathrm{gen}}$$

such that $d(x_i, x_{i+1}) = 1$ for all $1 \leq i < t$, and

$$\Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{x_t|x_{t-1}} \circ \cdots \circ \Theta^{\mathrm{norm}}_{x_2|x_1}.$$

Thus, to prove the proposition, it suffices to show that $\Theta^{\mathrm{norm}}_{y|x}$ is an isomorphism for all $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $d(x, y) = 1$, so we assume $d(x, y) = 1$. Let $H \subset \mathcal{A}_{x_0}$ denote the unique affine hyperplane in $\mathfrak{H}$ such that x and y are on opposite sides of H. If H is not $\mathcal{K}$-relevant, Lemma 3.5.11 implies that $\Theta^{\mathrm{norm}}_{y|x}$ is an isomorphism. Suppose that H is $\mathcal{K}$-relevant. We write $s = s_H$. Then there exists $x' \in \mathcal{A}_{\mathrm{gen}}$ such that $d(x, x') = d(y, sx') = 0$ and $d(x', sx') = 1$. By Lemma 3.5.12, we have

$$\Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{y|sx'} \circ \Theta^{\mathrm{norm}}_{sx'|x'} \circ \Theta^{\mathrm{norm}}_{x'|x}.$$

We already showed that the operators $\Theta^{\mathrm{norm}}_{x'|x}$ and $\Theta^{\mathrm{norm}}_{y|sx'}$ are isomorphisms. Moreover, since $c_{sx',s}$ is an isomorphism, it follows from Corollary 3.8.11 that the operator $\Theta^{\mathrm{norm}}_{sx'|x'} = (c_{sx',s})^{-1} \circ \Phi_{x',s}$ is also an isomorphism. Thus $\Theta^{\mathrm{norm}}_{y|x}$ is also an isomorphism. □

Corollary 3.8.13. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. For $x \in \mathcal{A}_{\mathrm{gen}}$ and $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$, the endomorphism $\Phi_{x,w}$ is invertible.*

Proof. The claim follows from the definition of $\Phi_{x,w}$ as $c_{w^{-1}x,w} \circ \Theta^{\mathrm{norm}}_{w^{-1}x|x}$, where $c_{w^{-1}x,w}$ is an isomorphism and Proposition 3.8.12. □

In order to show that the coefficient $p_{x,H}$ in the quadratic relation in Corollary 3.8.8 is non-zero and independent of x, we need a few more lemmas.

Lemma 3.8.14. *Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and let $x, x', y \in \mathcal{A}_{\mathrm{gen}}$. We assume that the points x and x' are on the same side of H and $\mathfrak{H}_{\mathcal{K}\text{-rel};x,y} = \{H\}$. Then we have*

$$\Theta^{\mathrm{norm}}_{x'|x} \circ \Theta^{\mathrm{norm}}_{x|y} = \Theta^{\mathrm{norm}}_{x'|y} \quad \text{and} \quad \Theta^{\mathrm{norm}}_{y|x} \circ \Theta^{\mathrm{norm}}_{x|x'} = \Theta^{\mathrm{norm}}_{y|x'}.$$

Proof. Since the points x and x' are on the same side of H and $\mathfrak{H}_{\mathcal{K}\text{-rel};x,y} = \{H\}$, we have $\mathfrak{H}_{\mathcal{K}\text{-rel};x,x'} \cap \mathfrak{H}_{\mathcal{K}\text{-rel};x,y} = \emptyset$. Then Lemma 3.5.14 implies that $d_{\mathcal{K}\text{-rel}}(x', x) + d_{\mathcal{K}\text{-rel}}(x, y) = d_{\mathcal{K}\text{-rel}}(x', y)$ and $d_{\mathcal{K}\text{-rel}}(y, x) + d_{\mathcal{K}\text{-rel}}(x, x') = d_{\mathcal{K}\text{-rel}}(y, x')$. Hence, the claim follows from Proposition 3.5.15. □

Lemma 3.8.15. *Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and let $x, x', y, y' \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{\mathcal{K}\text{-rel};x,y} = \mathfrak{H}_{\mathcal{K}\text{-rel};x',y'} = \{H\}$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{ind}_{K_x}^{G(F)}(\rho_x) & \xrightarrow{\Theta^{\mathrm{norm}}_{x'|x}} & \mathrm{ind}_{K_{x'}}^{G(F)}(\rho_{x'}) \\ {\scriptstyle \Theta^{\mathrm{norm}}_{y|x}} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \Theta^{\mathrm{norm}}_{y'|x'}} \\ \mathrm{ind}_{K_y}^{G(F)}(\rho_y) & \xrightarrow{\Theta^{\mathrm{norm}}_{y'|y}} & \mathrm{ind}_{K_{y'}}^{G(F)}(\rho_{y'}). \end{array} \tag{3.8.16}$$

Proof. First, we consider the case that x and x' are on the same side of H. Then our assumptions and Lemma 3.8.14 imply that

$$\Theta^{\mathrm{norm}}_{y'|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{y'|x} = \Theta^{\mathrm{norm}}_{y'|y} \circ \Theta^{\mathrm{norm}}_{y|x}.$$

Thus, we obtain the claim.

Next, we consider the case that x and x' are on opposite sides of H. In this case, the points x and y' are on the same side of H, and the points y and x' are on the same side of H. Then according to Lemma 3.8.14, we have

$$\Theta^{\mathrm{norm}}_{x'|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{x'|x} \quad \text{and} \quad \Theta^{\mathrm{norm}}_{y'|x'} \circ \Theta^{\mathrm{norm}}_{x'|y} = \Theta^{\mathrm{norm}}_{y'|y}. \tag{3.8.16a}$$

Combining the equations in (3.8.16a), we obtain that

$$\Theta^{\mathrm{norm}}_{y'|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{y'|x'} \circ \Theta^{\mathrm{norm}}_{x'|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{y'|y} \circ \Theta^{\mathrm{norm}}_{y|x}.$$

Thus, we also obtain the claim in this case. □

Now, we obtain the following corollary of Lemma 3.8.15, that will be used to show that the coefficient $p_{x,H}$ is independent of x.

Corollary 3.8.17. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that*

$$\mathfrak{H}_{\mathcal{K}\text{-rel};x,s_H x} = \mathfrak{H}_{\mathcal{K}\text{-rel};y,s_H y} = \{H\}.$$

Then we have

$$\Theta^{\mathrm{norm}}_{y|x} \circ \Phi_{x,s_H} \circ \left(\Theta^{\mathrm{norm}}_{y|x}\right)^{-1} = \Phi_{y,s_H}.$$

Proof. We write $s = s_H$ and recall that by Lemma 3.5.19 we have

$$\Theta^{\mathrm{norm}}_{y|x} \circ c_{s^{-1}x,s} = c_{s^{-1}y,s} \circ \Theta^{\mathrm{norm}}_{s^{-1}y|s^{-1}x}.$$

Using this identity, Proposition 3.8.12, and Lemma 3.8.15, we obtain

$$\begin{aligned}
\Theta^{\mathrm{norm}}_{y|x} \circ \Phi_{x,s} \circ \left(\Theta^{\mathrm{norm}}_{y|x}\right)^{-1} &= \Theta^{\mathrm{norm}}_{y|x} \circ \left(c_{s^{-1}x,s} \circ \Theta^{\mathrm{norm}}_{s^{-1}x|x}\right) \circ \left(\Theta^{\mathrm{norm}}_{y|x}\right)^{-1} \\
&= \left(\Theta^{\mathrm{norm}}_{y|x} \circ c_{s^{-1}x,s}\right) \circ \Theta^{\mathrm{norm}}_{s^{-1}x|x} \circ \left(\Theta^{\mathrm{norm}}_{y|x}\right)^{-1} \\
&= \left(c_{s^{-1}y,s} \circ \Theta^{\mathrm{norm}}_{s^{-1}y|s^{-1}x}\right) \circ \Theta^{\mathrm{norm}}_{s^{-1}x|x} \circ \left(\Theta^{\mathrm{norm}}_{y|x}\right)^{-1} \\
&= c_{s^{-1}y,s} \circ \left(\Theta^{\mathrm{norm}}_{s^{-1}y|s^{-1}x} \circ \Theta^{\mathrm{norm}}_{s^{-1}x|x} \circ \left(\Theta^{\mathrm{norm}}_{y|x}\right)^{-1}\right) \\
&= c_{s^{-1}y,s} \circ \Theta^{\mathrm{norm}}_{s^{-1}y|y} = \Phi_{y,s}.
\end{aligned}$$

□

By using Lemma 3.8.15, we can also prove the following proposition about $\mathcal{K}$-relevant hyperplanes.

Proposition 3.8.18. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and let $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{\mathcal{K}\text{-rel};x,y} = \{H\}$. Then we have*

$$\Theta_{x|y} \circ \Theta_{y|x} \notin \mathcal{C} \cdot \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)}.$$

Proof. Since $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, there exists $x', y' \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x',y'} = \{H\}$ and

$$\Theta_{x'|y'} \circ \Theta_{y'|x'} \notin \mathcal{C} \cdot \mathrm{id}_{\mathrm{ind}_{K_{x'}}^{G(F)}(\rho_{x'})},$$

or, equivalently,

$$\Theta^{\mathrm{norm}}_{x'|y'} \circ \Theta^{\mathrm{norm}}_{y'|x'} \notin \mathcal{C} \cdot \mathrm{id}_{\mathrm{ind}_{K_{x'}}^{G(F)}(\rho_{x'})}.$$

According to Lemma 3.8.15, we have

$$\Theta^{\mathrm{norm}}_{y'|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{y'|y} \circ \Theta^{\mathrm{norm}}_{y|x} \qquad \text{and} \qquad \Theta^{\mathrm{norm}}_{x'|y'} \circ \Theta^{\mathrm{norm}}_{y'|y} = \Theta^{\mathrm{norm}}_{x'|x} \circ \Theta^{\mathrm{norm}}_{x|y}.$$

Combining them, we obtain that

$$\Theta^{\mathrm{norm}}_{x'|y'} \circ \Theta^{\mathrm{norm}}_{y'|x'} \circ \Theta^{\mathrm{norm}}_{x'|x} = \Theta^{\mathrm{norm}}_{x'|y'} \circ \Theta^{\mathrm{norm}}_{y'|y} \circ \Theta^{\mathrm{norm}}_{y|x} = \Theta^{\mathrm{norm}}_{x'|x} \circ \Theta^{\mathrm{norm}}_{x|y} \circ \Theta^{\mathrm{norm}}_{y|x}.$$

Using Proposition 3.8.12, we have

$$\Theta^{\mathrm{norm}}_{x'|y'} \circ \Theta^{\mathrm{norm}}_{y'|x'} = \Theta^{\mathrm{norm}}_{x'|x} \circ \left(\Theta^{\mathrm{norm}}_{x|y} \circ \Theta^{\mathrm{norm}}_{y|x}\right) \circ \left(\Theta^{\mathrm{norm}}_{x'|x}\right)^{-1}.$$

Thus, we obtain that

$$\Theta^{\mathrm{norm}}_{x|y} \circ \Theta^{\mathrm{norm}}_{y|x} \notin \mathcal{C} \cdot \mathrm{id}_{\mathrm{ind}_{K_{x}}^{G(F)}(\rho_{x})},$$

and hence

$$\Theta_{x|y} \circ \Theta_{y|x} \notin \mathcal{C} \cdot \mathrm{id}_{\mathrm{ind}_{K_{x}}^{G(F)}(\rho_{x})}.$$

□

Now, we can strengthen the statement of Corollary 3.8.8.

Proposition 3.8.19. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{\mathcal{K}\text{-rel};x,s_H x} = \{H\}$. Then there exist non-zero $p_{x,H}, q_{x,H} \in \mathcal{C}$ such that*

$$(\Phi_{x,s_H})^2 = p_{x,H} \cdot \Phi_{x,s_H} + q_{x,H} \cdot \Phi_{x,1}.$$

Moreover, the coefficients $p_{x,H}$ and $q_{x,H}$ are independent of the point $x \in \mathcal{A}_{\mathrm{gen}}$ that satisfies $\mathfrak{H}_{\mathcal{K}\text{-rel};x,s_H x} = \{H\}$.

Proof. Let $y \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{y,s_H y} = \{H\}$. Then by Corollary 3.8.8 and Lemma 3.8.10 there exist $p_{y,H}, q_{y,H} \in \mathcal{C}$ such that $q_{y,H}$ is non-zero and

$$(\Phi_{y,s_H})^2 = p_{y,H} \cdot \Phi_{y,s_H} + q_{y,H} \cdot \Phi_{y,1}.$$

On the other hand, according to Corollary 3.8.17, we have

$$\Theta^{\mathrm{norm}}_{y|x} \circ \Phi_{x,s_H} \circ \left(\Theta^{\mathrm{norm}}_{y|x}\right)^{-1} = \Phi_{y,s_H}.$$

Hence, we also obtain that

$$(\Phi_{x,s_H})^2 = p_{y,H} \cdot \Phi_{x,s_H} + q_{y,H} \cdot \Phi_{x,1}.$$

Since $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $\mathfrak{H}_{\mathcal{K}\text{-rel};x,s_H x} = \{H\}$, by combining Proposition 3.8.18 with Lemma 3.8.9, we obtain that $(\Phi_{x,s_H})^2 \notin \mathcal{C} \cdot \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)}$, that is, $p_{y,H} \neq 0$. Moreover, $p_{y,H}$ and $q_{y,H}$ are independent of x. □

Scaling the operators Φ_{x,s_H} we can ensure that the coefficients in the above Proposition have a particularly nice form. In order to do so, we choose a partitioning of the coefficient field as follows.

Choice 3.8.20. Let $\mathcal{C}_{>1}$ be a subset of $\mathcal{C}^\times \smallsetminus \{1\}$ such that

$$\left|\{q, q^{-1}\} \cap \mathcal{C}_{>1}\right| = 1 \quad \text{for all} \quad q \in \mathcal{C}^\times \smallsetminus \{1\}.$$

This choice is necessary as the coefficient q_{s_H} in the quadratic relation in the Proposition below is a priori only determined up to taking its inverse, and we can freely choose any of the (at most) two options. Strictly speaking it suffices to only make such a choice for the subset of $\mathcal{C}$ that contains all possible values of q_{s_H}. In particular, if $\ell = 0$, then we will see below in Proposition 3.9.1 that q_{s_H} is a quotient of two positive integers, so we only need to choose a subset of the positive rational numbers. One possible choice in that case is to say that $q_{s_H} > 1$, which is the choice that, for example, Howlett and Lehrer [HL80, (3.19) Definition and (4.14) Theorem] and Morris [Mor93, §6.9 and 7.12. Theorem] took in their settings, and which is the reason for our choice of notation "$\mathcal{C}_{>1}$".

Proposition 3.8.21. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2. Then for $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{\mathcal{K}\text{-rel};x,s_Hx} = \{H\}$, there exist unique scalars $d_{s_H} \in \mathcal{C}^\times$ and $q_{s_H} \in \mathcal{C}_{>1}$ such that*

$$(d_{s_H} \cdot \Phi_{x,s_H})^2 = (q_{s_H} - 1) \cdot (d_{s_H} \cdot \Phi_{x,s_H}) + q_{s_H} \cdot \Phi_{x,1}.$$

Moreover, the scalars d_{s_H} and q_{s_H} do not depend on the choice of the point $x \in \mathcal{A}_{\mathrm{gen}}$.

Proof. According to Proposition 3.8.19, the endomorphism Φ_{x,s_H} satisfies the quadratic relation

$$(\Phi_{x,s_H})^2 = p_{x,H} \cdot \Phi_{x,s_H} + q_{x,H} \cdot \Phi_{x,1}$$

with $p_{x,H}$ and $q_{x,H}$ non-zero. Hence, for $d \in \mathcal{C}^\times$, the endomorphism $d \cdot \Phi_{x,s_H}$ satisfies the quadratic relation

$$(d \cdot \Phi_{x,s_H})^2 = p_{x,H} d \cdot (d \cdot \Phi_{x,s_H}) + q_{x,H} d^2 \cdot \Phi_{x,1}.$$

Thus, to prove the first claim of the proposition, it suffices to show that the quadratic equation

$$q_{x,H} d^2 - p_{x,H} d - 1 = 0 \tag{3.8.21a}$$

has a unique solution $d = d_{s_H}$ with $q_{x,H} d_{s_H}^2 \in \mathcal{C}_{>1}$. Since $q_{x,H} \neq 0$ and $\mathcal{C}$ is algebraically closed, Equation (3.8.21a) has two solutions $d_1, d_2 \in \mathcal{C}$ that are possibly equal. Note that $d = 0$ is not a solution of Equation (3.8.21a), i.e., $d_1, d_2 \in \mathcal{C}^\times$. Let $i = 1$ or 2. Since $q_{x,H}, d_i \in \mathcal{C}^\times$, we have $q_{x,H} d_i^2 \neq 0$. Moreover, since $p_{x,H} \neq 0$, we have $q_{x,H} d_i^2 - 1 = p_{x,H} d_i \neq 0$, that is, $q_{x,H} d_i^2 \neq 1$. Hence, we conclude that $q_{x,H} d_i^2 \in \mathcal{C}^\times \smallsetminus \{1\}$. Since the solutions d_1 and d_2 satisfy $d_1 d_2 = -1/q_{x,H}$, we have $q_{x,H} d_2^2 = (q_{x,H} d_1^2)^{-1}$. Thus, our choice of $\mathcal{C}_{>1}$ in Choice 3.8.20 implies that

$$\left|\left\{q_{x,H} d_1^2, q_{x,H} d_2^2\right\} \cap \mathcal{C}_{>1}\right| = 1.$$

If $q_{x,H} d_1^2 \neq q_{x,H} d_2^2$, this equation implies that exactly one of the solutions d_1 and d_2 satisfies $q_{x,H} d_i^2 \in \mathcal{C}_{>1}$, as desired.

It remains to consider the case that $q_{x,H} d_1^2 = q_{x,H} d_2^2$. Then we have

$$p_{x,H} d_1 = (q_{x,H} d_1^2 - 1) = (q_{x,H} d_2^2 - 1) = p_{x,H} d_2,$$

so $d_1 = d_2$. We also note that since $q_{x,H}d_1^2 \neq 1$ and $q_{x,H}d_1^2 = q_{x,H}d_2^2 = (q_{x,H}d_1^2)^{-1}$, we obtain that the characteristic ℓ of $\mathcal{C}$ cannot be two in this case, and we have $q_{x,H}d_1^2 = q_{x,H}d_2^2 = -1$. Thus, we conclude that Equation (3.8.21a) has the unique solution $d_{s_H} \coloneqq d_1 = d_2$, and we have $q_{x,H}d_{s_H}^2 = -1 \in \mathcal{C}_{>1}$.

Since the coefficients $p_{x,H}$ and $q_{x,H}$ are independent of the point $x \in \mathcal{A}_{\text{gen}}$, Equation (3.8.21a) is independent of the point $x \in \mathcal{A}_{\text{gen}}$. Hence, the scalars d_{s_H} and $q_{s_H} = q_{x,H}d_{s_H}^2$ are also independent of the point $x \in \mathcal{A}_{\text{gen}}$. □

Proposition 3.8.21 allows us to refine Choice 3.5.17 by replacing $T_{\widetilde{s}}$ in Choice 3.5.17 by $d_s \cdot T_{\widetilde{s}}$ for all lifts $\widetilde{s} \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ of $s \in S_{\mathcal{K}\text{-rel}}$. Then noting Remark 3.5.25, we obtain a choice of $\mathcal{T}$ that satisfies the following properties.

Choice 3.8.22. We choose a family of non-zero elements

$$\mathcal{T} = \{T_n \in \operatorname{Hom}_{K_M}({}^n\rho_M, \rho_M)\}_{n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}}$$

satisfying Conditions (1) and (2) of Choice 3.5.17 and such that for every $s \in S_{\mathcal{K}\text{-rel}}$ there exists $q_s \in \mathcal{C}_{>1}$ such that

$$(\Phi_s)^2 = (q_s - 1) \cdot \Phi_s + q_s \cdot \Phi_1.$$

Proposition 3.8.23. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2 and fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20. Then we can choose $\mathcal{T}$ to satisfy all the properties in Choice 3.8.22. This means in particular that for each $s \in S_{\mathcal{K}\text{-rel}}$, there exists $q_s \in \mathcal{C}_{>1}$ such that the element Φ_s satisfies the quadratic relation*

$$(\Phi_s)^2 = (q_s - 1) \cdot \Phi_s + q_s \cdot \Phi_1.$$

Moreover, if $s, s' \in S_{\mathcal{K}\text{-rel}}$ are $W(\rho_M)^{\heartsuit}_{[x_0]_M}$-conjugate, then we have $q_s = q_{s'}$.

Proof. Since we have $\mathfrak{H}_{\mathcal{K}\text{-rel};x_0,sx_0} = \{H_s\}$ for all $s \in S_{\mathcal{K}\text{-rel}}$, the claim that we can choose $\mathcal{T}$ as in Choice 3.8.22 follows from Proposition 3.8.21 as explained in the paragraph before Choice 3.8.22 and the quadratic relation is part of the properties stated in Choice 3.8.22. It remains to show that we have $q_s = q_{s'}$ if $s, s' \in S_{\mathcal{K}\text{-rel}}$ are $W(\rho_M)^{\heartsuit}_{[x_0]_M}$-conjugate. Let $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ such that $wsw^{-1} = s'$. By replacing w with ws if necessary, we may suppose that

$$\ell_{\mathcal{K}\text{-rel}}(ws) = \ell_{\mathcal{K}\text{-rel}}(w) + 1 = \ell_{\mathcal{K}\text{-rel}}(w) + \ell_{\mathcal{K}\text{-rel}}(s).$$

Then, according to Proposition 3.6.4, we have

$$\Phi_w \Phi_s = \mu^{\mathcal{T}}(w, s) \cdot \Phi_{ws}.$$

On the other hand, we have

$$\ell_{\mathcal{K}\text{-rel}}(s'w) = \ell_{\mathcal{K}\text{-rel}}(ws) = \ell_{\mathcal{K}\text{-rel}}(w) + 1 = \ell_{\mathcal{K}\text{-rel}}(s') + \ell_{\mathcal{K}\text{-rel}}(w).$$

Hence, Proposition 3.6.4 also implies that

$$\Phi_{s'}\Phi_w = \mu^{\mathcal{T}}(s', w) \cdot \Phi_{s'w} = \mu^{\mathcal{T}}(s', w) \cdot \Phi_{ws}.$$

Writing $d_{s,s'} = \mu^{\mathcal{T}}(w,s)/\mu^{\mathcal{T}}(s',w) \in \mathcal{C}^\times$ and using that Φ_w is invertible by Corollary 3.8.13, we obtain by combining the above equations that $\Phi_w \Phi_s \Phi_w^{-1} = d_{s,s'} \cdot \Phi_{s'}$. Since $\Phi_w \Phi_s \Phi_w^{-1}$ satisfies the same quadratic relation as Φ_s, we obtain

$$\left(d_{s,s'} \cdot \Phi_{s'}\right)^2 = (q_s - 1) \cdot \left(d_{s,s'} \cdot \Phi_{s'}\right) + q_s \cdot \Phi_1,$$

On the other hand, we have the quadratic relation

$$(\Phi_{s'})^2 = (q_{s'} - 1) \cdot \Phi_{s'} + q_{s'} \cdot \Phi_1.$$

Since $q_s, q_{s'} \in \mathcal{C}_{>1}$, Proposition 3.8.21 implies that $d_{s,s'} = 1$ and $q_s = q_{s'}$. □

Remark 3.8.24. The proof of Proposition 3.8.23 implies the following claim. Suppose that $s, s' \in S_{\mathcal{K}\text{-rel}}$ and $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ satisfy $wsw^{-1} = s'$ and $\ell_{\mathcal{K}\text{-rel}}(ws) = \ell_{\mathcal{K}\text{-rel}}(w) + \ell_{\mathcal{K}\text{-rel}}(s)$. Then we have $\Phi_w \Phi_s \Phi_w^{-1} = \Phi_{s'}$. In particular, for all $s \in S_{\mathcal{K}\text{-rel}}$ and $t \in \Omega(\rho_M)$, we have $\Phi_t \Phi_s \Phi_t^{-1} = \Phi_{tst^{-1}}$.

3.9 The coefficients in the quadratic relations in characteristic zero

In this subsection, we keep the notation from the previous subsection. In addition, we assume that the characteristic ℓ of $\mathcal{C}$ is zero or a banal prime for $G(F)$. Recall that a *banal* prime is one that does not divide the order of any finite quotient of any compact, open subgroup of $G(F)$. This assumption allows us to give in Proposition 3.9.1 below a more explicit description of the coefficients q_s in the quadratic relations in Proposition 3.8.23, as it implies that for every finite quotient H of a compact, open subgroup of $G(F)$, the finite-dimensional $\mathcal{C}$-representations of H are semisimple. Moreover, the order $|H|$ of H is not zero in $\mathcal{C}$ and neither is the dimension of any irreducible $\mathcal{C}$-representation of H. Thus, we can divide by these numbers wherever convenient.

Proposition 3.9.1. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20, and choose $\mathcal{T}$ as in Choice 3.8.22.*

Let $s \in S_{\mathcal{K}\text{-rel}}$ and let $x \in \mathcal{A}_{\text{gen}}$ such that $\mathfrak{H}_{x,sx} = \{H_s\}$. Then the compactly induced representation $\text{ind}_{K_x}^{K_{x,s}}(\rho_x)$ decomposes into a direct sum of two inequivalent irreducible representations:

$$\text{ind}_{K_x}^{K_{x,s}}(\rho_x) = \rho_1 \oplus \rho_2.$$

We assume without loss of generality that ρ_1 and ρ_2 are chosen so that $\dim_{\mathcal{C}}(\rho_1)/\dim_{\mathcal{C}}(\rho_2) \in \mathcal{C}_{>1} \cup \{1\}$. Then

$$q_s = \frac{\dim_{\mathcal{C}}(\rho_1)}{\dim_{\mathcal{C}}(\rho_2)}.$$

To prove Proposition 3.9.1, we present a generalization of a result of Howlett and Lehrer [HL80].

Lemma 3.9.2. *Let H be a compact topological group and K an open subgroup of H. If $\ell \neq 0$, then we suppose that the order of no finite quotient of H is divisible by ℓ. Let (ρ, V_ρ) be an irreducible smooth representation of K. We suppose that the induced representation $\text{ind}_K^H(\rho)$ decomposes into a direct sum of two inequivalent irreducible representations ρ_1 and ρ_2.*

Then there exists an element Φ_h of $\text{End}_H\left(\text{ind}_K^H(\rho)\right) \simeq \mathcal{H}(H,\rho)$ such that $\text{supp}(\Phi_h) = KhK$ with $h \in I_H(\rho) \smallsetminus K$, and such that we have

$$(\Phi_h)^2 = (q - 1) \cdot \Phi_h + q \cdot \Phi_1,$$

where Φ_1 denotes the identity map on $\text{ind}_K^H(V_\rho)$ and $q = \frac{\dim_{\mathcal{C}}(\rho_1)}{\dim_{\mathcal{C}}(\rho_2)}$.

Here we define the Hecke algebra $\mathcal{H}(H, \rho)$ associated to (K, ρ) analogously to Section 2.2.

Proof of Lemma 3.9.2. The proof of the lemma is essentially the same as [HL80, Theorem 3.18 (ii)], but we include it here for the convenience of the reader.

Since ρ is a smooth representation of a compact group K, the kernel of ρ is a normal, open subgroup of K. We define the normal, open subgroup N of H by

$$N = \bigcap_{a \in H/K} a(\ker \rho)a^{-1}.$$

By replacing H and K with H/N and K/N, respectively, and regarding ρ as a representation of K/N, we may suppose that H and K are finite groups. Moreover, by our assumption on the order of the finite subquotients of H, if $\ell \neq 0$, then $|H|$ and $|K|$ are coprime to ℓ, and hence $\dim_{\mathcal{C}}(\rho_1)$, $\dim_{\mathcal{C}}(\rho_2)$, and $\dim_{\mathcal{C}}(\rho)$ are also coprime to ℓ.

We fix a non-zero element $\Phi'_h \in \mathrm{End}_H\left(\mathrm{ind}_K^H(\rho)\right)$ such that $\mathrm{supp}(\Phi'_h) = KhK$. Since $\mathrm{End}_H\left(\mathrm{ind}_K^H(\rho)\right)$ is two dimensional by Schur's lemma, Φ_1 and Φ'_h give a basis of $\mathrm{End}_H\left(\mathrm{ind}_K^H(\rho)\right)$. We denote by $p_1 \in \mathrm{End}_H\left(\mathrm{ind}_K^H(\rho)\right)$ the projection onto ρ_1 with respect to the decomposition $\mathrm{ind}_K^H(\rho) = \rho_1 \oplus \rho_2$, and we write

$$p_1 = \lambda \cdot \Phi_1 + \mu \cdot \Phi'_h \tag{3.9.2a}$$

for some $\lambda, \mu \in \mathcal{C}$. Since the element p_1 satisfies $p_1^2 = p_1$, Equation (3.9.2a) implies that

$$\left(\lambda \cdot \Phi_1 + \mu \cdot \Phi'_h\right)^2 = \lambda \cdot \Phi_1 + \mu \cdot \Phi'_h.$$

Hence, we have

$$\left(\mu \cdot \Phi'_h\right)^2 = (1 - 2\lambda) \cdot \left(\mu \cdot \Phi'_h\right) + \lambda(1 - \lambda) \cdot \Phi_1. \tag{3.9.2b}$$

We can calculate the value λ as follows. We fix a non-zero element $v \in V_\rho$ and define the element $f_v \in \mathrm{ind}_K^H(V_\rho)$ by

$$f_v(h') = \begin{cases} \rho(h')(v) & (h' \in K), \\ 0 & (\text{otherwise}) \end{cases}$$

for $h' \in H$. We will apply both sides of (3.9.2a) to f_v and compare the values at the identity element of H. Recall that the projection p_1 can be written as

$$p_1 = \frac{\dim_{\mathcal{C}}(\rho_1)}{|H|} \sum_{h' \in H} \mathrm{tr}(\rho_1(h'^{-1})) \cdot \left(\mathrm{ind}_K^H(\rho)\right)(h'),$$

where $\mathrm{tr}(\rho_1(h'^{-1}))$ denotes the trace of the linear map $\rho_1(h'^{-1})$.

Hence, we have

$$\begin{aligned}(p_1(f_v))(1) &= \frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}\sum_{h\in H}\operatorname{tr}(\rho_1(h'^{-1}))\cdot\left(\left(\operatorname{ind}_K^H(\rho)\right)(h')\cdot(f_v)\right)(1)\\ &= \frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}\sum_{h'\in H}\operatorname{tr}(\rho_1(h'^{-1}))\cdot f_v(h')\\ &= \frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}\sum_{k\in K}\operatorname{tr}(\rho_1(k^{-1}))\cdot\rho(k)(v)\\ &= \left(\frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}\sum_{k\in K}\operatorname{tr}(\rho_1(k^{-1}))\cdot\rho(k)\right)(v).\end{aligned}$$

Since

$$\frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}\sum_{k\in K}\operatorname{tr}(\rho_1(k^{-1}))\cdot\rho(k)\in\operatorname{End}_K(\rho),$$

and ρ is an irreducible representation of K, Schur's lemma implies that

$$\frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}\sum_{k\in K}\operatorname{tr}(\rho_1(k^{-1}))\cdot\rho(k)$$

is a scalar multiplication on V_ρ. Moreover, letting $(\ ,\ )_H$ and $(\ ,\ )_K$ denote the inner products of finite-dimensional representations of H and K, the scalar is calculated as

$$\begin{aligned}\frac{1}{\dim_{\mathcal{C}}(\rho)}\frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}\sum_{k\in K}\operatorname{tr}(\rho_1(k^{-1}))\cdot\operatorname{tr}(\rho(k)) &= \frac{1}{\dim_{\mathcal{C}}(\rho)}\frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}|K|\,(\rho_1|_K,\rho)_K\\ &= \frac{1}{\dim_{\mathcal{C}}(\rho)}\frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}|K|\,(\rho_1,\operatorname{ind}_K^H(\rho))_H\\ &= \frac{1}{\dim_{\mathcal{C}}(\rho)}\frac{\dim_{\mathcal{C}}(\rho_1)}{|H|}|K|\\ &= \dim_{\mathcal{C}}(\rho_1)\cdot(\dim_{\mathcal{C}}(\rho)\cdot|H/K|)^{-1}\\ &= \frac{\dim_{\mathcal{C}}(\rho_1)}{\dim_{\mathcal{C}}\left(\operatorname{ind}_K^H(\rho)\right)}\\ &= \frac{\dim_{\mathcal{C}}(\rho_1)}{\dim_{\mathcal{C}}(\rho_1)+\dim_{\mathcal{C}}(\rho_2)}.\end{aligned}$$

On the other hand, the right-hand side of (3.9.2a) is calculated as

$$\left(\left(\lambda\cdot\Phi_1+\mu\cdot\Phi'_h\right)(f_v)\right)(1)=\lambda\cdot v.$$

Thus, we have

$$\frac{\dim_{\mathcal{C}}(\rho_1)}{\dim_{\mathcal{C}}(\rho_1)+\dim_{\mathcal{C}}(\rho_2)}=\lambda. \tag{3.9.2c}$$

Now, we define the element Φ_h of $\mathrm{End}_H\left(\mathrm{ind}_K^H(\rho)\right)$ by

$$\Phi_h = -\frac{\dim_{\mathcal{C}}(\rho_1)+\dim_{\mathcal{C}}(\rho_2)}{\dim_{\mathcal{C}}(\rho_2)} \cdot \mu \cdot \Phi'_h.$$

Note that Φ_h is non-zero, because $\lambda(1-\lambda) \neq 0$ implies by (3.9.2b) that μ is non-zero. Then according to (3.9.2b) and (3.9.2c), we have

$$(\Phi_h)^2 = (q-1)\cdot \Phi_h + q \cdot \Phi_1. \qquad \square$$

Proof of Proposition 3.9.1. Since the group $K_{x,s}$ is compact, the representation $\mathrm{ind}_{K_x}^{K_{x,s}}(\rho_x)$ is semi-simple. Hence, the first claim follows from Lemma 3.8.7. To prove the second claim, note that by Proposition 3.8.21, the elements $\Phi_s = \Phi_{x_0,s}$ and $\Phi_{x,s}$ satisfy the same quadratic relation. Hence, we have

$$(\Phi_{x,s})^2 = (q_s - 1)\cdot \Phi_{x,s} + q_s \cdot \Phi_{x,1}. \tag{3.9.1a}$$

On the other hand, applying Lemma 3.9.2 to $H = K_{x,s}$, $K = K_x$, and $\rho = \rho_x$, we obtain that there exists an element $\Phi \in \mathcal{H}(K_{x,s}, \rho_x)_s$ such that

$$(\Phi)^2 = (q-1)\cdot \Phi + q \cdot \Phi_1, \tag{3.9.1b}$$

where $q = \frac{\dim_{\mathcal{C}}(\rho_1)}{\dim_{\mathcal{C}}(\rho_2)}$. Since $\mathcal{H}(K_{x,s}, \rho_x)_s$ is one dimensional by Proposition 3.4.18, there exists $d \in \mathcal{C}^\times$ such that $\Phi = d \cdot \Phi_{x,s}$. Then comparing (3.9.1a) with (3.9.1b), we have $q \neq 1$, and replacing ρ_1 with ρ_2 if necessary, we may suppose that $q \in \mathcal{C}_{>1}$. Then the proposition follows from Proposition 3.8.21 and Equations (3.9.1a) and (3.9.1b). $\square$

3.10 The description of the Hecke algebra

In this subsection, we will prove our main theorem of this section, Theorem 3.10.10, about the structure of the Hecke algebra $\mathcal{H}(G(F), \rho_{x_0})$. We keep the notation from Section 3.8, i.e., the notation from all previous subsections, but allowing the coefficient field $\mathcal{C}$ again to be of any characteristic other than p. We also assume that all previous axioms hold, i.e., Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, and fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20.

Recall that we defined the elements $\Phi_w \in \mathrm{End}_{G(F)}\left(\mathrm{ind}_{K_x}^{G(F)}(\rho_x)\right) \simeq \mathcal{H}(G(F), \rho_x)$ for all $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ in Definition 3.5.23 and that these endomorphisms depend on the choice of a family $\mathcal{T}$, Choice 3.8.22). In this subsection, we will adjust $\mathcal{T}$, and thus $\{\Phi_w\}$, in a way that makes the latter more compatible with the group structure of $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$, see Choice 3.10.3 below, so that we can use these basis elements to write down an explicit isomorphism between the Hecke algebra $\mathcal{H}(G(F), \rho_{x_0})$ and a semi-direct product of an affine Hecke algebra with a twisted group algebra in Theorem 3.10.10. In order to obtain the subalgebra isomorphic to the affine Hecke algebra, we start by proving appropriate braid relations.

Lemma 3.10.1. *Let s_1 and s_2 be distinct elements of $S_{\mathcal{K}\text{-rel}}$ such that the order of $s_1 s_2$ in $W(\rho_M)_{\mathrm{aff}}$ is $m < \infty$. Then we have*

$$\underbrace{\Phi_{s_1}\Phi_{s_2}\Phi_{s_1}\cdots}_{m\text{ terms}} = \underbrace{\Phi_{s_2}\Phi_{s_1}\Phi_{s_2}\cdots}_{m\text{ terms}}.$$

Proof. According to Proposition 3.6.4, there exists $c \in \mathcal{C}^\times$ such that

$$\underbrace{\Phi_{s_1}\Phi_{s_2}\Phi_{s_1}\cdots}_{m\,\text{terms}} = c \cdot \underbrace{\Phi_{s_2}\Phi_{s_1}\Phi_{s_2}\cdots}_{m\,\text{terms}}.$$

We will prove that $c = 1$. We write

$$n := \begin{cases} 1 & (2 \mid m) \\ 2 & (2 \nmid m) \end{cases} \quad \text{and} \quad \Phi_{2,1}. := \underbrace{\Phi_{s_2}\Phi_{s_1}\Phi_{s_2}\cdots}_{m-1\,\text{terms}} \quad \text{so that} \quad \Phi_{s_1}\Phi_{2,1} = c \cdot \Phi_{2,1}\Phi_{s_n}.$$

Then we obtain

$$(\Phi_{s_1})^2\,\Phi_{2,1} = \Phi_{s_1} \circ c \cdot \Phi_{2,1}\Phi_{s_n} = c^2 \cdot \Phi_{2,1}\,(\Phi_{s_n})^2\,.$$

Since Φ_{s_1} and Φ_{s_n} satisfy the quadratic relations

$$(\Phi_{s_1})^2 = (q_{s_1} - 1)\cdot\Phi_{s_1} + q_{s_1}\cdot\Phi_1 \quad \text{and} \quad (\Phi_{s_n})^2 = (q_{s_n} - 1)\cdot\Phi_{s_n} + q_{s_n}\cdot\Phi_1,$$

we obtain that

$$\begin{aligned}(q_{s_1} - 1)\cdot\Phi_{s_1}\Phi_{2,1} + q_{s_1}\cdot\Phi_{2,1} &= c^2\cdot((q_{s_n} - 1)\cdot\Phi_{2,1}\Phi_{s_n} + q_{s_n}\cdot\Phi_{2,1}) \\ &= c\cdot(q_{s_n} - 1)\cdot\Phi_{s_1}\Phi_{2,1} + c^2\cdot q_{s_n}\cdot\Phi_{2,1}.\end{aligned}$$

Using that $\Phi_{s_1}\Phi_{2,1}$ and $\Phi_{2,1}$ are linearly independent, we conclude that

$$q_{s_1} - 1 = c\cdot(q_{s_n} - 1). \tag{3.10.1a}$$

Since $s_n = \Big(\underbrace{s_2s_1s_2\cdots}_{m-1\,\text{terms}}\Big)^{-1} s_1 \Big(\underbrace{s_2s_1s_2\cdots}_{m-1\,\text{terms}}\Big)$, the element s_n is $W(\rho_M)^{\heartsuit}_{[x_0]_M}$-conjugate to s_1, and therefore by Proposition 3.8.23 we have $q_{s_1} = q_{s_n} \neq 1$. Thus Equation (3.10.1a) implies that $c = 1$. □

According to Proposition 3.7.4, the group $W(\rho_M)_{\text{aff}} \simeq W_{\mathcal{K}\text{-rel}}$ is a Coxeter group. Hence, we can define the notion of a reduced expression for $w \in W(\rho_M)_{\text{aff}}$ in the usual way.

Corollary 3.10.2. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, 3.8.2, and that $\mathcal{T}$ is chosen as in Choice 3.8.22. Let $w \in W(\rho_M)_{\text{aff}}$. Then the element $\Phi_{s_1}\Phi_{s_2}\cdots\Phi_{s_r}$ does not depend on the choice of a reduced expression $w = s_1s_2\cdots s_r$ for w.*

Proof. According to [BB05, Theorem 3.3.1 (ii)], every two reduced expressions for w can be connected via a sequence of braid-moves. Then the corollary follows from Lemma 3.10.1. □

We now refine Choice 3.8.22. Let $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\text{aff}} \smallsetminus \{1\}$ with reduced expression $w = s_1s_2\cdots s_r$. According to Proposition 3.6.4, there exists $d_{tw} \in \mathcal{C}^\times$ such that $d_{tw}\cdot\Phi_{tw} = \Phi_t\Phi_{s_1}\Phi_{s_2}\cdots\Phi_{s_r}$. According to Corollary 3.10.2, the scalar d_{tw} does not depend on the choice of a reduced expression for w. By replacing T_n with $d_{tw}\cdot T_n$ for all lifts n of tw in $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ and noting Remark 3.5.25, we obtain the following refinement of Choice 3.8.22:

Choice 3.10.3. We choose a family of non-zero elements

$$\mathcal{T} = \{T_n \in \operatorname{Hom}_{K_M}({}^n\rho_M, \rho_M)\}_{n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}}$$

that satisfies the following conditions:

(1) We have $T_1 = \mathrm{id}_{\rho_M}$.

(2) For all $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ and $k \in K_M \cap N(\rho_M)^{\heartsuit}_{[x_0]_M}$, we have $T_{nk} = T_n \circ \rho_M(k)$.

(3) For every $s \in S_{\mathcal{K}\text{-rel}}$, there exists $q_s \in \mathcal{C}_{>1}$ such that $(\Phi_s)^2 = (q_s - 1) \cdot \Phi_s + q_s \cdot \Phi_1$.

(4) For all $w \in W(\rho_M)_{\mathrm{aff}}$ with reduced expression $w = s_1 s_2 \cdots s_r$, we have $\Phi_w = \Phi_{s_1}\Phi_{s_2}\cdots\Phi_{s_r}$.

(5) For all $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$, we have $\Phi_t\Phi_w = \Phi_{tw}$.

Proposition 3.10.4. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2 and fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20. Then we can choose $\mathcal{T}$ to satisfy all the properties in Choice 3.10.3.*

Proof. According to Proposition 3.8.23, we can choose $\mathcal{T}$ as in Choice 3.8.22 and the paragraph before Choice 3.10.3 explains a rescaling that yields a choice $\mathcal{T}$ satisfying all the desired properties. □

Remark 3.10.5. According to Proposition 3.8.21, Condition (3) of Choice 3.10.3 determines the choices of $T_{\widetilde{s}} \in \mathcal{T}$ for all lifts $\widetilde{s} \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ of $s \in S_{\mathcal{K}\text{-rel}}$. Condition (4) then determines the choices of $T_n \in \mathcal{T}$ for all lifts $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ of $w \in W(\rho_M)_{\mathrm{aff}}$. On the other hand, we can choose $T_n \in \mathcal{T}$ freely for a set of representatives of lifts $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ of $t \in \Omega(\rho_M) \smallsetminus \{1\}$. Once we fix $T_n \in \mathcal{T}$ for such a set of representatives, Conditions (1) through (5) determine all elements of $\mathcal{T}$. We will make a special choice of $T_n \in \mathcal{T}$ for lifts $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ of $t \in \Omega(\rho_M) \smallsetminus \{1\}$ in Choice 3.11.5 below.

We also note the following lemma:

Lemma 3.10.6. *We choose $\mathcal{T}$ as in Choice 3.10.3. Then we have $\Phi_t\Phi_w = \Phi_{twt^{-1}}\Phi_t$ for all $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$.*

Proof. We will prove that $\Phi_{twt^{-1}} = \Phi_t\Phi_w\Phi_t^{-1}$. Since we have $\Phi_w = \Phi_{s_1}\Phi_{s_2}\cdots\Phi_{s_r}$ for a reduced expression $w = s_1 s_2 \cdots s_r$, we may suppose that $w = s \in S_{\mathcal{K}\text{-rel}}$. Now, the lemma follows from Remark 3.8.24. □

We can rewrite Conditions (4) and (5) of Choice 3.10.3 and Lemma 3.10.6 in terms of the 2-cocycle $\mu^{\mathcal{T}}$.

Proposition 3.10.7. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, and we fix $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20 and $\mathcal{T}$ as in Choice 3.10.3. Then for $v, w \in W(\rho_M)_{\mathrm{aff}}$ such that $\ell_{\mathcal{K}\text{-rel}}(vw) = \ell_{\mathcal{K}\text{-rel}}(v) + \ell_{\mathcal{K}\text{-rel}}(w)$, we have $\mu^{\mathcal{T}}(v, w) = 1$. In addition, for $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$, we have $\mu^{\mathcal{T}}(t, w) = \mu^{\mathcal{T}}(w, t) = 1$.*

Proof. The first claim follows from Proposition 3.6.4 and Condition (4) of Choice 3.10.3. The second claim follows by combining Proposition 3.6.4 with Condition (5) of Choice 3.10.3 and Lemma 3.10.6 and recalling that $\Omega(\rho_M)$ normalizes $W(\rho_M)_{\mathrm{aff}}$. $\square$

Notation 3.10.8. We recall several standard notations.

(a) We denote by $\mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ the affine Hecke algebra with $\mathcal{C}$-coefficients associated with the affine Weyl group $W(\rho_M)_{\mathrm{aff}}$, its set of generators $S_{\mathcal{K}\text{-rel}}$, and the parameter function q. This means $\mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ is a $\mathcal{C}$-algebra with a vector-space basis

$$\{\mathbb{T}_w \mid w \in W(\rho_M)_{\mathrm{aff}}\},$$

and relations generated by

$$\mathbb{T}_s^2 = (q_s - 1) \cdot \mathbb{T}_s + q_s \cdot \mathbb{T}_1 \quad \text{and} \quad \mathbb{T}_w = \mathbb{T}_{s_1}\mathbb{T}_{s_2} \cdots \mathbb{T}_{s_r} \tag{3.10.9}$$

for all $s \in S_{\mathcal{K}\text{-rel}}$ and all $w \in W(\rho_M)_{\mathrm{aff}}$ with a reduced expression $w = s_1 s_2 \cdots s_r$.

(b) The twisted group algebra $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}]$ is defined to be the vector space $\bigoplus_{t\in\Omega(\rho_M)} \mathcal{C} \cdot b_t$ equipped with the multiplication $b_{t_1} \cdot b_{t_2} = \mu^{\mathcal{T}}(t_1, t_2) \cdot b_{t_1 t_2}$ for $t_1, t_2 \in \Omega(\rho_M)$.

(c) We define the $\mathcal{C}$-algebra $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ to be the vector space

$$\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \otimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$$

with multiplication rules given by:

(1) $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}]$ and $\mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ are embedded as subalgebras,

(2) for $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$, we have $b_t \cdot \mathbb{T}_w = \mathbb{T}_{twt^{-1}} \cdot b_t$.

Now we obtain the following structure theorem for our Hecke algebra:

Theorem 3.10.10. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, and fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20. Then we can choose $\mathcal{T}$ as in Choice 3.10.3, and the resulting $\mathcal{C}$-linear map*

$$\mathcal{I}(\rho_{x_0}) \colon \mathcal{H}(G(F), \rho_{x_0}) \to \mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q),$$

defined by

$$\mathcal{I}(\rho_{x_0})(\varphi_{tw}) = b_t \cdot \mathbb{T}_w \quad (t \in \Omega(\rho_M), w \in W(\rho_M)_{\mathrm{aff}})$$

is an isomorphism of $\mathcal{C}$-algebras, where $\mu^{\mathcal{T}}$ denotes the restriction to $\Omega(\rho_M) \times \Omega(\rho_M)$ of the 2-cocycle introduced in Notation 3.6.1 and q denotes the parameter function $s \mapsto q_s$ appearing in Choice 3.10.3(3).

Proof. By Proposition 3.10.4 we can choose $\mathcal{T}$ as in Choice 3.10.3. According to Corollary 3.5.27 and Proposition 3.7.6, $\mathcal{I}(\rho_{x_0})$ is an isomorphism of vector spaces. Moreover, by Proposition 3.6.4 the restriction of $\mathcal{I}(\rho_{x_0})$ to $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}]$ is an algebra homomorphism, and Conditions (3) and (4) of Choice 3.10.3 imply that the restriction of $\mathcal{I}(\rho_{x_0})$ to $\mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ is also an algebra homomorphism. Combining these observations with Lemma 3.10.6, we obtain that $\mathcal{I}(\rho_{x_0})$ is an isomorphism of $\mathcal{C}$-algebras. $\square$

3.11 Anti-involution of the Hecke algebra

In this subsection, we keep the notation from the previous subsection, and moreover we assume that $\mathcal{C}$ admits a nontrivial involution $c \mapsto \bar{c}$. The fixed field of this involution will then be a real closed field that we will denote by $\mathcal{R}$. (See Corollary VI.9.3, Theorem IX.2.2, and Proposition IX.2.4 of [Lan02]. For instance, $\mathcal{R} = \mathbb{R}$ if $\mathcal{C} = \mathbb{C}$ and $c \mapsto \bar{c}$ denotes complex conjugation. Such an involution exists if and only if $\mathcal{C}$ has characteristic zero. In this case, one can construct the fixed field $\mathcal{R}$, and thus the involution, by choosing a maximal, totally ordered subfield of $\mathcal{C}$. Note that the choice of $\mathcal{R}$, and thus the choice of involution, is never unique. However, in practice, the number of explicit choices that one can make (i.e., choices that do not require the Axiom of Choice), is usually zero (say, for $\mathcal{C} = \overline{\mathbb{Q}}_\ell$) or one (say, for $\mathcal{C} = \mathbb{C}$ or $\overline{\mathbb{Q}}$).

Having made such a choice, write $|c| = c\bar{c}$ for $c \in \mathcal{C}$, and call $\bar{c}$ the *$\mathcal{C}$-conjugate* of c (which depends on our fixed choice of involution even though it is not reflected in the notation). For a vector space V over $\mathcal{C}$, call a function $\langle\ ,\ \rangle\colon V \times V \to \mathcal{C}$ *Hermitian* if it is linear in the first variable, $\mathcal{C}$-conjugate-linear in the second variable, and we have $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in V$.

Assume that all previous axioms hold, i.e., Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, and fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20, and $\mathcal{T}$ as in Choice 3.10.3. Following [Ciu18, Section 4.4], we define a support-inverting, $\mathcal{C}$-conjugate-linear anti-involution $\varphi \mapsto \varphi^*$ on $\mathcal{H}(G(F), \rho_{x_0})$ as follows. We fix a K_{x_0}-invariant, positive-definite Hermitian form $\langle\ ,\ \rangle_{\rho_{x_0}}$ on $V_{\rho_{x_0}}$. Such a form exists because K_{x_0} is compact, and it is unique up to $\mathcal{R}$-scalar multiples because ρ_{x_0} is irreducible. For $\varphi \in \mathcal{H}(G(F), \rho_{x_0})$, we define $\varphi^* \in \mathcal{H}(G(F), \rho_{x_0})$ by

$$\langle \varphi^*(g)(v), w\rangle_{\rho_{x_0}} = \langle v, \varphi(g^{-1})(w)\rangle_{\rho_{x_0}} \qquad \text{for all } g \in G(F) \text{ and } v, w \in V_{\rho_{x_0}}. \tag{3.11.1}$$

We note that the map $\varphi \mapsto \varphi^*$ is a support-inverting, $\mathcal{C}$-conjugate-linear anti-involution that does not depend on the choice of Hermitian form on $V_{\rho_{x_0}}$. Hence, for each $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$, there is some scalar $c_w \in \mathcal{C}^\times$ such that $\varphi_w^* = c_w \varphi_{w^{-1}}$.

Lemma 3.11.2. *For all $w \in W(\rho_M)^{\heartsuit}_{[x_0]_M}$ of order 2, we have $|c_w| = 1$. Moreover, if $w \in S_{\mathcal{K}\text{-rel}}$, then $c_w = 1$.*

Proof. If $w^2 = 1$, then we have $\varphi_w = \varphi_w^{**} = (c_w\varphi_w)^* = \bar{c}_w(\varphi_w)^* = \bar{c}_w c_w \varphi_w$, so $\bar{c}_w c_w = 1$, as required. Suppose that $w \in S_{\mathcal{K}\text{-rel}}$. Since $(\ )^*$ is an anti-involution, the element $\varphi_w^* = c_w\varphi_w$ satisfies the same quadratic relation as φ_w. Thus, Proposition 3.8.21 implies that $c_w = 1$. □

Corollary 3.11.3. *We have $c_w = 1$ for all $w \in W(\rho_M)_{\mathrm{aff}}$.*

Proof. Let $w \in W(\rho_M)_{\mathrm{aff}}$ with reduced expression $w = s_1 s_2 \cdots s_r$. According to Condition (4) in Choice 3.10.3, we have $\varphi_w = \varphi_{s_1}\varphi_{s_2}\cdots\varphi_{s_r}$ and $\varphi_{w^{-1}} = \varphi_{s_r}\varphi_{s_{r-1}}\cdots\varphi_{s_1}$. Thus, we obtain that

$$\varphi_w^* = (\varphi_{s_1}\varphi_{s_2}\cdots\varphi_{s_r})^* = \varphi_{s_r}^*\varphi_{s_{r-1}}^*\cdots\varphi_{s_1}^* = \varphi_{s_r}\varphi_{s_{r-1}}\cdots\varphi_{s_1} = \varphi_{w^{-1}},$$

as required. □

Proposition 3.11.4. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20, and choose $\mathcal{T}$ as in Choice 3.10.3. Then we can choose a set of scalars $\{d_t \in \mathcal{C}^\times \mid t \in \Omega(\rho_M)\}$ such that $(d_t\varphi_t)^* = d_{t^{-1}}\varphi_{t^{-1}}$ for all $t \in \Omega(\rho_M)$.*

Proof. Let $d_1 = 1$. For all nontrivial $t \in \Omega(\rho_M)$ of order 2, from Lemma 3.11.2 and Hilbert's Theorem 90, we can choose $d_t \in \mathcal{C}^\times$ such that $d_t/\bar{d}_t = c_t$. Then $(d_t\varphi_t)^* = \bar{d}_t c_t \varphi_t = d_t\varphi_t, = d_{t^{-1}}\varphi_{t^{-1}}$, as required.

Now consider the set of elements $t \in \Omega(\rho_M)$ such that t^2 is nontrivial. Partition this set into pairs $\{t, t^{-1}\}$. Given a pair, let us choose an element arbitrarily, and without loss of generality call it t. Let $d_t = 1$ and $d_{t^{-1}} = c_t$. Then we have $(d_t\varphi_t)^* = \varphi_t^* = c_t\varphi_{t^{-1}} = d_{t^{-1}}\varphi_{t^{-1}}$ and $(d_{t^{-1}}\varphi_{t^{-1}})^* = (c_t\varphi_{t^{-1}})^* = \varphi_t^{**} = \varphi_t = d_t\varphi_t$, as required. □

This allows us to refine our choice of $\{T_n\}$ as follows, see Proposition 3.11.6.

Choice 3.11.5. We choose a family of non-zero elements

$$\mathcal{T} = \{T_n \in \operatorname{Hom}_{K_M}({}^n\rho_M, \rho_M)\}_{n\in N(\rho_M)^{\heartsuit}_{[x_0]_M}}$$

that satisfies all the conditions in Choice 3.10.3, and for all $t \in \Omega(\rho_M)$, we have $\varphi_t^* = \varphi_{t^{-1}}$.

Proposition 3.11.6. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2 and fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20. Then we can choose $\mathcal{T}$ to satisfy all the properties in Choice 3.11.5.*

Proof. The proposition follows from Remark 3.10.5 by replacing the previous choice of T_n with $d_t{\cdot}T_n$ for all lifts $n \in N(\rho_M)^{\heartsuit}_{[x_0]_M}$ of $t \in \Omega(\rho_M)$, and Proposition 3.11.4 (see also Remark 3.5.25). □

We define a $\mathcal{C}$-conjugate-linear map $(\ \ )^*$ on $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ by

$$(b_t \cdot \mathbb{T}_w)^* = b_{t^{-1}} \cdot \mathbb{T}_{tw^{-1}t^{-1}} = \mathbb{T}_{w^{-1}} \cdot b_{t^{-1}}$$

for $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$.

Proposition 3.11.7. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2 and fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20. Then we can choose $\mathcal{T}$ as in Choice 3.11.5, and the resulting $\mathcal{C}$-algebra isomorphism*

$$\mathcal{I}(\rho_{x_0})\colon \mathcal{H}(G(F), \rho_{x_0}) \to \mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$$

in Theorem 3.10.10 preserves the anti-involutions on both sides. In particular, the $\mathcal{C}$-conjugate-linear map $(\ \ )^*$ on $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ is an anti-involution.

Proof. By Proposition 3.11.6 we can choose $\mathcal{T}$ as in Choice 3.11.5. The claim that $\mathcal{I}(\rho_{x_0})$ preserves the anti-involutions follows from Corollary 3.11.3, the condition $\varphi_t^* = \varphi_{t^{-1}}$ for $t \in \Omega(\rho_M)$ in Choice 3.11.5, and the anti-involution property. Since $\mathcal{I}(\rho_{x_0})$ is an isomorphism of $\mathcal{C}$-algebras, we obtain the last claim. □

3.12 (Non)uniqueness of the Hecke algebra isomorphism

In this subsection, we keep the notation from Section 3.10. Hence, we allow the coefficient field $\mathcal{C}$ again to be any algebraically closed field of characteristic $\ell \neq p$. We also assume that all previous axioms hold, i.e., Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20, and choose $\mathcal{T}$ as in Choice 3.10.3. We can make explicit how much choice we had in constructing the particular isomorphism in Theorem 3.10.10.

We say that a $\mathcal{C}$-linear bijection Ψ on $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ is *support-preserving* if for all $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$, we have $\Psi(b_t \cdot \mathbb{T}_w) = c\, b_t \cdot \mathbb{T}_w$ for some scalar $c \in \mathcal{C}^\times$ that might depend on t and w. We also say that a $\mathcal{C}$-linear bijection $\mathcal{I}\colon \mathcal{H}(G(F), \rho_{x_0}) \xrightarrow{\sim} \mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ is support-preserving if for all $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$, we have $\mathcal{I}(\varphi_{tw}) = c\, b_t \cdot \mathbb{T}_w$ for some scalar $c \in \mathcal{C}^\times$. For $\chi \in \mathrm{Hom}_{\mathbb{Z}}(\Omega(\rho_M), \mathcal{C}^\times)$, we define the support-preserving algebra automorphism Ψ_χ of $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ by $\Psi_\chi(b_t \cdot \mathbb{T}_w) = \chi(t) b_t \cdot \mathbb{T}_w$ for $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$.

Proposition 3.12.1. *A $\mathcal{C}$-linear bijection Ψ on $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$ is a support-preserving algebra automorphism if and only if there exists some $\chi \in \mathrm{Hom}_{\mathbb{Z}}(\Omega(\rho_M), \mathcal{C}^\times)$ such that $\Psi = \Psi_\chi$.*

If $\mathcal{C}$ admits a nontrivial involution $c \mapsto \bar{c}$ and $\mathcal{T}$ satisfies all the properties in Choice 3.11.5, then the isomorphism Ψ_χ preserves the anti-involution defined in Section 3.11 if and only if $|\chi(t)| = 1$ for all $t \in \Omega(\rho_M)$.

Proof. Suppose that Ψ is a support-preserving algebra automorphism of $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$. Since Ψ preserves support, for all $t \in \Omega(\rho_M)$ and $w \in W(\rho_M)_{\mathrm{aff}}$ we must have that $\Psi(b_t \cdot \mathbb{T}_w) = \chi(tw) b_t \cdot \mathbb{T}_w$ for some scalar $\chi(tw) \in \mathcal{C}^\times$. Since Ψ is an algebra isomorphism, we must have that $\chi(tw) = \chi(t)\chi(w)$. Using the first relation of (3.10.9), we have

$$\chi(s)^2((q_s - 1) \cdot \mathbb{T}_s + q_s \cdot \mathbb{T}_1) = (\chi(s)\mathbb{T}_s)^2 = \Psi(\mathbb{T}_s)^2 = \Psi(\mathbb{T}_s^2) = (q_s - 1) \cdot (\chi(s)\mathbb{T}_s) + q_s \cdot \mathbb{T}_1$$

for all $s \in S_{\mathcal{K}\text{-rel}}$. Comparing the coefficients of $\mathbb{T}_s$ and using $q_s \in \mathcal{C}^\times \smallsetminus \{1\}$, we obtain that $\chi(s) = 1$ for all $s \in S_{\mathcal{K}\text{-rel}}$. Then the second relation in (3.10.9) and the fact that Ψ is an algebra isomorphism imply that $\chi(w) = 1$ for all $w \in W(\rho_M)_{\mathrm{aff}}$. Thus, Ψ acts via the identity on $\mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$, and it is determined by the scalars $\chi(t)$ for $t \in \Omega(\rho_M)$. The condition of Ψ preserving the multiplication of $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}]$ is equivalent to χ being a homomorphism.

The last claim follows from the definitions of Ψ_χ and the anti-involution $(\ )^*$ on $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$. □

Note that the group $\mathrm{Hom}_{\mathbb{Z}}(\Omega(\rho_M), \mathcal{C}^\times)$ acts on the group of support-preserving algebra automorphisms via $\psi : \Psi_\chi \longmapsto \Psi_{\psi\chi}$.

Proposition 3.12.2. *We assume Axioms 3.4.1, 3.4.3, 3.7.1, and 3.8.2, we fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20, and we choose a family $\mathcal{T}$ as in Choice 3.10.3. Then the set of support-preserving $\mathcal{C}$-algebra isomorphisms*

$$\mathcal{H}(G(F), \rho_{x_0}) \xrightarrow{\sim} \mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q),$$

is a torsor under $\mathrm{Hom}_{\mathbb{Z}}(\Omega(\rho_M), \mathcal{C}^\times)$.

If $\mathcal{C}$ admits a nontrivial involution $c \mapsto \bar{c}$ and $\mathcal{T}$ satisfies all the properties in Choice 3.11.5, then the set of such isomorphisms that also preserve the anti-involutions defined in Section 3.11 is a torsor under the group $\mathrm{Hom}_{\mathbb{Z}}(\Omega(\rho_M), \{z \in \mathcal{C}^\times \mid |z| = 1\})$.

Proof. Fix such an isomorphism $\mathcal{I}$, which exists by Theorem 3.10.10. For every $\chi \in \mathrm{Hom}_{\mathbb{Z}}(\Omega(\rho_M), \mathcal{C}^\times)$, we have a support-preserving $\mathcal{C}$-algebra automorphism Ψ_χ of $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$, as in the proof of Proposition 3.12.1. Let $\mathcal{I}_\chi$ denote the composition $\Psi_\chi \circ \mathcal{I}$. Then it is clear that this is a support-preserving algebra isomorphism from $\mathcal{H}(G(F), \rho_{x_0})$ to $\mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q)$, and that all such isomorphisms arise in this way.

Suppose that $\mathcal{C}$ admits a nontrivial involution $c \mapsto \bar{c}$ and $\mathcal{T}$ satisfies all the properties in Choice 3.11.5. If we also assume that $\mathcal{I}$ preserves the anti-involution (such an isomorphism exists by Proposition 3.11.7), then the last part of Proposition 3.12.1 implies that $\mathcal{I}_\chi$ preserves the anti-involution if and only if $\chi \in \mathrm{Hom}_{\mathbb{Z}}(\Omega(\rho_M), \{z \in \mathcal{C}^\times \mid |z| = 1\})$. ☐

4 Comparison of Hecke algebras

We have seen in Section 3 that if a pair (K, ρ) consisting of a compact, open subgroup K of $G(F)$ and an irreducible representation ρ of K satisfies some axioms, then we can determine the structure of the attached Hecke algebra $\mathcal{H}(G(F), \rho)$. Now we consider a pair (K^0, ρ^0) for a subgroup G^0 of G. We will show that if the two pairs (K, ρ) and (K^0, ρ^0) are related according to some axioms, Axioms 4.1.2, 4.2.1, and 4.3.1, then we have a support-preserving algebra isomorphism $\mathcal{H}(G^0(F), \rho^0) \longrightarrow \mathcal{H}(G(F), \rho)$, see Theorem 4.4.8 and Theorem 4.4.11.

As a special case, in [FOAM, Theorem 4.3.11], we will obtain an isomorphism between the Hecke algebra attached to a type of a Bernstein block of arbitrary depth constructed by Kim and Yu and the Hecke algebra of a depth-zero type. Readers interested in this case might find it helpful to first read Sections 4.1 and 4.2 of [FOAM] to have an example in mind for the objects appearing in the axiomatic set-up below. A reader solely interested in the setting of [FOAM] might also completely replace all the below objects by those introduced in [FOAM, Section 4] with the same symbols.

4.1 The set-up

We let G^0 be a connected reductive subgroup of G of same rank as G and let $j\colon \mathcal{B}(G^0, F) \hookrightarrow \mathcal{B}(G, F)$ be an admissible embedding of enlarged Bruhat–Tits buildings in the sense of [KP23, §14.2]. We note that an admissible embedding exists by [KP23, Proposition 14.6.1, Corollary 14.7.3, Proposition 14.8.4]. For example, in the setting of types constructed by Kim and Yu, G^0 is a twisted Levi subgroup of G, see [FOAM, Definition 4.1.1].

Let M^0 be a Levi subgroup of G^0 for which $A_{M^0} = A_M$, where M denotes the centralizer $Z_G(A_{M^0})$ of A_{M^0} in G. Recall that A_{M^0}, respectively, A_M, denotes the maximal split torus in the center of M^0, respectively M. Note that M is a Levi subgroup of G and we have $G^0 \cap M = M^0$. We fix a commutative diagram

$$\begin{array}{ccc} \mathcal{B}(G^0, F) & \xrightarrow{\ j\ } & \mathcal{B}(G, F) \\ \uparrow & & \uparrow \\ \mathcal{B}(M^0, F) & \longrightarrow & \mathcal{B}(M, F) \end{array}$$

of admissible embeddings of buildings and identify $\mathcal{B}(M^0, F)$, $\mathcal{B}(M, F)$, and $\mathcal{B}(G^0, F)$ with their images in $\mathcal{B}(G, F)$. Let $x_0 \in \mathcal{B}(M^0, F)$. Then the affine spaces $x_0 + (X_*(A_{M^0}) \otimes_{\mathbb{Z}} \mathbb{R})$ and $x_0 + (X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R})$ attached to (G^0, M^0, x_0) and (G, M, x_0) in Section 3.1 agree, so we may denote both by the same symbol $\mathcal{A}_{x_0}$ without ambiguity. By our definition of M, the subgroups $N_{G^0}(M^0)(F)_{[x_0]_{M^0}}$ of $G^0(F)$ and $N_G(M)(F)_{[x_0]_M}$ of $G(F)$ introduced in Section 3.1 satisfy the relation

$$N_{G^0}(M^0)(F)_{[x_0]_{M^0}} = G^0(F) \cap N_G(M)(F)_{[x_0]_M}. \tag{4.1.1}$$

We fix a locally finite set of affine hyperplanes $\mathfrak{H}$ in $\mathcal{A}_{x_0}$ that do not contain x_0. We let K_{M^0}, resp., K_M, be a compact, open subgroup of $M^0(F)_{x_0}$, resp., $M(F)_{x_0}$, and $(\rho_{M^0}, V_{\rho_{M^0}})$, resp., (ρ_M, V_{ρ_M}), be an irreducible smooth representation of K_{M^0}, resp., K_M. Let $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ be a subgroup of $N(\rho_{M^0})_{[x_0]_{M^0}}$ containing $A_{M^0}(F)$. For example, in [FOAM], we take $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} = N(\rho_{M^0})_{[x_0]_{M^0}}$, and (K_{M^0}, ρ_{M^0}) and (K_M, ρ_M) are depth-zero and positive-depth supercuspidal types as constructed by Yu ([Yu01]) if $\mathcal{C} = \mathbb{C}$.

We will assume the following axiom that relates the general pairs (K_{M^0}, ρ_{M^0}) and (K_M, ρ_M).

Axiom 4.1.2.

(1) There exists a compact, open subgroup $K_{M,0+}$ of $M(F)$ that is normalized by $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cdot K_{M^0}$ such that the group $M^0(F) \cap K_{M,0+}$ is contained in the kernel of ρ_{M^0}, and we have $K_M = K_{M^0} \cdot K_{M,0+}$.

(2) There exists an irreducible smooth representation (κ_M, V_{κ_M}) of K_M that extends to a smooth representation $\widetilde{\kappa}_M$ of the group

$$\widetilde{K}_M := N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cdot K_M$$

such that

$$\rho_M \simeq \inf(\rho_{M^0}) \otimes \kappa_M,$$

where $\inf(\rho_{M^0})$ is the inflation of ρ_{M^0} to K_M via the surjection

$$K_M = K_{M^0} \cdot K_{M,0+} \twoheadrightarrow (K_{M^0} \cdot K_{M,0+})/K_{M,0+} \simeq K_{M^0}/\left(K_{M^0} \cap K_{M,0+}\right).$$

Remark 4.1.3. In the setting of [FOAM], the group $K_{M,0+}$ is defined in [FOAM, (4.1.3)], κ_M is a twist of a Weil–Heisenberg representation defined in [FOAM, Section 4.1, p. 42], and the extension $\widetilde{\kappa}_M$ of κ_M is constructed explicitly in [FOAM, Definition 4.3.3] and the preceding text. That Axiom 4.1.2 holds in this setting is proven in [FOAM, Proposition 4.3.4]. The existence of an extension $\widetilde{\kappa}_M$ of κ_M as required in this axiom was the key challenge in proving that all our axioms are satisfied in the setting of [FOAM].

Remark 4.1.4. Axiom 4.1.2(1) in particular includes the statement that $M^0(F) \cap K_{M,0+} \subset K_{M^0}$, and hence the requirement $K_M = K_{M^0} \cdot K_{M,0+}$ of the axiom implies that $M^0(F) \cap K_M = K_{M^0}$. Since $G^0 \cap M = M^0$ and $K_M \subset M(F)$, we also have $G^0(F) \cap K_M = K_{M^0}$.

It will be convenient for us to choose the subgroup $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ of $N(\rho_M)_{[x_0]_M}$ as $N(\rho_M)^{\heartsuit}_{[x_0]_M} = N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$, but in order to do so, we first need the following lemma.

Lemma 4.1.5. *We have*

$$N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \subset N(\rho_M)_{[x_0]_M}.$$

Proof. Let $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$. According to Equation (4.1.1), we have $n \in N_G(M)(F)_{[x_0]_M}$. Hence, it suffices to prove that $n \in N_{G(F)}(\rho_M)$. According to Axiom 4.1.2(1), the element n normalizes the group K_M. We will prove that n normalizes the representation ρ_M. Let $T^0_n \colon {}^n\rho_{M^0} \xrightarrow{\sim} \rho_{M^0}$ be an isomorphism. Then the morphism

$$T^0_n \otimes \widetilde{\kappa}_M(n) : \ {}^n\rho_M = {}^n\rho_{M^0} \otimes {}^n\kappa_M \xrightarrow{\sim} \rho_{M^0} \otimes \kappa_M = \rho_M$$

is an isomorphism as well, hence $n \in N_{G(F)}(\rho_M)$. □

4.2 Compatible families of quasi-G-covers

We keep the notation and assumption from the previous subsection, i.e., Axiom 4.1.2 holds. In this subsection, we will define the quasi-G^0-cover $(K^0_{x_0}, \rho^0_{x_0})$ of (K_{M^0}, ρ_{M^0}) and the quasi-G-cover (K_{x_0}, ρ_{x_0}) of (K_M, ρ_M) to which we attach the Hecke algebras that we prove are isomorphic in Theorem 4.4.8. For example, if $\mathcal{C} = \mathbb{C}$, we can take (K_{x_0}, ρ_{x_0}) to be a type for a Bernstein block constructed by Kim and Yu as a G-cover of a supercuspidal type (K_M, ρ_M). In this case, the pair $(K^0_{x_0}, \rho^0_{x_0})$ can be taken as the twist by a quadratic character introduced in [FKS23] of the depth-zero type included in the input for the construction of (K_{x_0}, ρ_{x_0}). (See [FOAM, Section 4.1] for details, where we include the twist by the quadratic character in the construction of (K_{x_0}, ρ_{x_0}) and therefore can take $(K^0_{x_0}, \rho^0_{x_0})$ as the depth-zero type input for this twisted construction.) As in Section 3.4, we will define not only one quasi-cover, but a whole family of quasi-covers. Moreover, we will formulate a compatibility condition between the family of quasi-G^0-covers and the family of quasi-G-covers, see Axiom 4.2.1.

Let

$$\mathcal{K}^0 = \left\{ (K^0_x, K^0_{x,+}, (\rho^0_x, V_{\rho^0_x})) \right\}_{x \in \mathcal{A}_{\mathrm{gen}}}$$

be a family of quasi-G^0-cover-candidates that satisfies Axiom 3.4.1 with the subgroup $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of $N(\rho_{M^0})_{[x_0]_{M^0}}$. According to Lemma 4.1.5, $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ is also a subgroup of $N(\rho_M)_{[x_0]_M}$. Let

$$\mathcal{K} = \{ (K_x, K_{x,+}, (\rho_x, V_{\rho_x})) \}_{x \in \mathcal{A}_{\mathrm{gen}}}$$

be a family of quasi-G-cover-candidates that satisfies Axiom 3.4.1 with the subgroup $N(\rho_M)^{\heartsuit}_{[x_0]_M} := N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of $N(\rho_M)_{[x_0]_M}$. Examples for such families are presented in the paragraph after [FOAM, Lemma 4.1.9] on p. 43 of [FOAM]. We assume that the two families satisfy the following compatibility properties.

Axiom 4.2.1. For each $x \in \mathcal{A}_{\mathrm{gen}}$, the following properties hold.

(1) There exists a compact, open subgroup $K_{x,0+}$ of K_x that is normalized by K^0_x such that the group $G^0(F) \cap K_{x,0+}$ is contained in the kernel of ρ^0_x, and we have $K_x = K^0_x \cdot K_{x,0+}$.

(2) There exists an irreducible smooth representation (κ_x, V_{κ_x}) of K_x such that

$$\rho_x = \inf\left(\rho_x^0\right) \otimes \kappa_x,$$

where $\inf\left(\rho_x^0\right)$ is the inflation of ρ_x^0 to K_x via the surjection

$$K_x = K_x^0 \cdot K_{x,0+} \twoheadrightarrow K_x^0 \cdot K_{x,0+}/K_{x,0+} \simeq K_x^0/\left(K_x^0 \cap K_{x,0+}\right).$$

(3) We have

$$I_{G(F)}(\rho_x) = K_x \cdot I_{G^0(F)}(\rho_x^0) \cdot K_x.$$

This axiom is verified for the setting of [FOAM] in [FOAM, Proposition 4.3.5].

4.3 $\mathcal{K}$-relevance vs. $\mathcal{K}^0$-relevance

We keep the notation and assumptions, i.e., Axioms 3.4.1, 4.1.2, and 4.2.1, from the previous subsection. Recall that we defined in Section 3.5 intertwining operators $\Theta_{y|x}$ for $x, y \in \mathcal{A}_{\mathrm{gen}}$. In order to distinguish the intertwining operators attached to G and $\mathcal{K}$ from the intertwining operators attached to G^0 and $\mathcal{K}^0$ we will denote the former by

$$\Theta_{y|x} \colon \operatorname{ind}_{K_x}^{G(F)}(\rho_x) \to \operatorname{ind}_{K_y}^{G(F)}(\rho_y)$$

and the latter by

$$\Theta_{y|x}^0 \colon \operatorname{ind}_{K_x^0}^{G^0(F)}(\rho_x^0) \to \operatorname{ind}_{K_y^0}^{G^0(F)}(\rho_y^0).$$

This means that an affine hyperplane $H \in \mathfrak{H}$ is called $\mathcal{K}^0$-relevant if there exists $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$ and $\Theta_{x|y}^0 \circ \Theta_{y|x}^0 \notin \mathcal{C} \cdot \mathrm{id}_{\operatorname{ind}_{K_x^0}^{G^0(F)}(\rho_x^0)}$. In this subsection, we will exhibit the relation between $\Theta_{y|x}$ and $\Theta_{y|x}^0$, see Lemma 4.3.6, and show that the notions of $\mathcal{K}$-relevant and $\mathcal{K}^0$-relevant coincide, see Corollary 4.3.7, under the assumption of the following axiom that ensures a compatibility between the data $\{K_x^0, K_{x,0+}, \kappa_x\}$ for different $x \in \mathcal{A}_{\mathrm{gen}}$.

Axiom 4.3.1. For any $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $d(x, y) = 1$, there exists a compact, open subgroup $K_{x,y}^0$ of $G^0(F)$, a compact, open subgroup $K_{x,y;0+}$ of $G(F)$, and an irreducible smooth representation $(\kappa_{x,y}, V_{\kappa_{x,y}})$ of $K_{x,y} \coloneqq K_{x,y}^0 \cdot K_{x,y;0+}$ such that

(1) $K_{x,y}^0$ contains K_x^0 and K_y^0.

(2) $K_{x,y;0+}$ is normalized by the group $K_{x,y}^0$, and we have

$$K_{x,0+} \subset \left(G^0(F) \cap K_{x,0+}\right) \cdot K_{x,y;0+} \quad \text{and} \quad K_{y,0+} \subset \left(G^0(F) \cap K_{y,0+}\right) \cdot K_{x,y;0+}.$$

(3) The group $G^0(F) \cap K_{x,y;0+}$ is contained in the kernels of ρ_x^0 and ρ_y^0.

(4) The restriction of $\kappa_{x,y}$ to $K_{x,y;0+}$ is irreducible.

(5) We have isomorphisms

$$\kappa_{x,y}|_{K_x^0 \cdot K_{x,y;0+}} \xrightarrow{\sim} \operatorname{ind}_{K_x}^{K_x^0 \cdot K_{x,y;0+}}(\kappa_x) \quad \text{and} \quad \kappa_{x,y}|_{K_y^0 \cdot K_{x,y;0+}} \xrightarrow{\sim} \operatorname{ind}_{K_y}^{K_y^0 \cdot K_{x,y;0+}}(\kappa_y).$$

Since the conditions of Axiom 4.3.1 are symmetric with respect to x and y, we may and do assume that $K^0_{x,y} = K^0_{y,x}$, $K_{x,y;0+} = K_{y,x;0+}$, and $\kappa_{x,y} = \kappa_{y,x}$.

In Section 4.4 below, where we will also assume Axioms 3.7.1 and 3.8.2, $K^0_{x,sx}$ and $K_{x,sx}$ will play the role of $K'_{x,s}$ in Axiom 3.8.2.

Remark 4.3.2. In the setting of [FOAM], the objects $K^0_{x,y}$, $K_{x,y;0+}$ and $\kappa_{x,y}$ are defined in [FOAM, Notation 4.3.6], and [FOAM, Lemma 4.3.7] shows that these objects satisfy Axiom 4.3.1 for the families $\mathcal{K}^0$ and $\mathcal{K}$ considered in [FOAM]. Axiom 4.3.1(5) is a reason why we need to twist the construction of Kim and Yu ([KY17]) in [FOAM] by a quadratic character following [FKS23]. Without this twist, the Hecke algebra isomorphism in [FOAM, Theorem 4.3.11], which is Theorem 4.4.8 applied in that setting, would not be true in general, see [FOAM, Section A.2] for an example.

Remark 4.3.3. Axiom 4.3.1(1) and (2) imply that

$$K_x \subset K^0_x \cdot K_{x,y;0+} \subset K_{x,y} \quad \text{and} \quad K_y \subset K^0_y \cdot K_{x,y;0+} \subset K_{x,y}.$$

We assume Axiom 4.3.1 from now on. Let $H \in \mathfrak{H}$ and $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$. Let $(K^0_{x,y}, K_{x,y;0+}, \kappa_{x,y})$ be the triple from Axiom 4.3.1. Since the representation ρ^0_x is trivial on the group $K^0_x \cap K_{x,y;0+} \subset G^0(F) \cap K_{x,y;0+}$, we can inflate ρ^0_x to the group $K^0_x \cdot K_{x,y;0+}$ via the surjection

$$K^0_x \cdot K_{x,y;0+} \longrightarrow K^0_x \cdot K_{x,y;0+}/K_{x,y;0+} \simeq K^0_x/\left(K^0_x \cap K_{x,y;0+}\right).$$

We use the same notation $\inf(\rho^0_x)$ for this inflation as for the inflation of ρ^0_x to K_x, which is just the restriction of the former to K_x. Then, by using Axiom 4.3.1(5), we have

$$\begin{aligned}
\operatorname{ind}_{K_x}^{K_{x,y}}(\rho_x) &= \operatorname{ind}_{K_x}^{K_{x,y}}\left(\inf\left(\rho^0_x\right) \otimes \kappa_x\right) & &\simeq \operatorname{ind}_{K^0_x \cdot K_{x,y;0+}}^{K_{x,y}}\left(\operatorname{ind}_{K_x}^{K^0_x \cdot K_{x,y;0+}}\left(\inf\left(\rho^0_x\right) \otimes \kappa_x\right)\right) \\
&\simeq \operatorname{ind}_{K^0_x \cdot K_{x,y;0+}}^{K_{x,y}}\left(\inf\left(\rho^0_x\right) \otimes \operatorname{ind}_{K_x}^{K^0_x \cdot K_{x,y;0+}} \kappa_x\right) & &\simeq \operatorname{ind}_{K^0_x \cdot K_{x,y;0+}}^{K_{x,y}}\left(\inf(\rho^0_x) \otimes \kappa_{x,y}|_{K^0_x \cdot K_{x,y;0+}}\right) \\
&\simeq \operatorname{ind}_{K^0_x \cdot K_{x,y;0+}}^{K_{x,y}}\left(\inf(\rho^0_x)\right) \otimes \kappa_{x,y} & &= \operatorname{ind}_{K^0_x \cdot K_{x,y;0+}}^{K^0_{x,y} \cdot K_{x,y;0+}}\left(\inf(\rho^0_x)\right) \otimes \kappa_{x,y} \\
&\simeq \inf\left(\operatorname{ind}_{K^0_x}^{K^0_{x,y}}(\rho^0_x)\right) \otimes \kappa_{x,y},
\end{aligned}$$

where $\inf\left(\operatorname{ind}_{K^0_x}^{K^0_{x,y}}(\rho^0_x)\right)$ denotes the inflation of the representation $\operatorname{ind}_{K^0_x}^{K^0_{x,y}}(\rho^0_x)$ to $K_{x,y}$ via the surjection

$$K_{x,y} = K^0_{x,y} \cdot K_{x,y;0+} \longrightarrow K^0_{x,y} \cdot K_{x,y;0+}/K_{x,y;0+} \simeq K^0_{x,y}/\left(K^0_{x,y} \cap K_{x,y;0+}\right).$$

We write this isomorphism as

$$I^{x,y}_x\colon \operatorname{ind}_{K_x}^{K_{x,y}}(\rho_x) \xrightarrow{\sim} \inf\left(\operatorname{ind}_{K^0_x}^{K^0_{x,y}}(\rho^0_x)\right) \otimes \kappa_{x,y}.$$

Similarly, we obtain an isomorphism

$$I^{x,y}_y\colon \operatorname{ind}_{K_y}^{K_{x,y}}(\rho_y) \xrightarrow{\sim} \inf\left(\operatorname{ind}_{K^0_y}^{K^0_{x,y}}(\rho^0_y)\right) \otimes \kappa_{x,y}.$$

According to Axiom 4.3.1(1), Remark 4.3.3, and the definitions of $\Theta_{y|x}$ and $\Theta^0_{y|x}$, we have

$$\Theta_{y|x} \in \operatorname{Hom}_{G(F)}\left(\operatorname{ind}_{K_x}^{G(F)}(\rho_x), \operatorname{ind}_{K_y}^{G(F)}(\rho_y)\right)_{K_{x,y}} \simeq \operatorname{Hom}_{K_{x,y}}\left(\operatorname{ind}_{K_x}^{K_{x,y}}(\rho_x), \operatorname{ind}_{K_y}^{K_{x,y}}(\rho_y)\right) \tag{4.3.4}$$

$$\Theta^0_{y|x} \in \mathrm{Hom}_{G^0(F)}\big(\mathrm{ind}_{K^0_x}^{G(F)}(\rho^0_x), \mathrm{ind}_{K^0_y}^{G(F)}(\rho^0_y)\big)_{K^0_{x,y}} \simeq \mathrm{Hom}_{K^0_{x,y}}\big(\mathrm{ind}_{K^0_x}^{K^0_{x,y}}(\rho^0_x), \mathrm{ind}_{K^0_y}^{K^0_{x,y}}(\rho^0_y)\big) \qquad (4.3.5)$$

(see Lemma 2.2.8) and therefore we may view $\Theta_{y|x}$ and $\Theta^0_{y|x}$ as elements of the latter spaces. Hence, we obtain an element

$$I^{x,y}_y \circ \Theta_{y|x} \circ (I^{x,y}_x)^{-1} \in \mathrm{Hom}_{K_{x,y}}\left(\mathrm{inf}\big(\mathrm{ind}_{K^0_x}^{K^0_{x,y}}(\rho^0_x)\big) \otimes \kappa_{x,y}, \mathrm{inf}\big(\mathrm{ind}_{K^0_y}^{K^0_{x,y}}(\rho^0_y)\big) \otimes \kappa_{x,y}\right).$$

Lemma 4.3.6. *There exists $c \in \mathcal{C}^\times$ such that*

$$I^{x,y}_y \circ \Theta_{y|x} \circ (I^{x,y}_x)^{-1} = c \cdot \Theta^0_{y|x} \otimes \mathrm{id}_{V_{\kappa_{x,y}}}.$$

Proof. Since $K_{x,y;0+}$ is contained in the kernels of the two representations $\mathrm{inf}\big(\mathrm{ind}_{K^0_x}^{K^0_{x,y}}(\rho^0_x)\big)$ and $\mathrm{inf}\big(\mathrm{ind}_{K^0_y}^{K^0_{x,y}}(\rho^0_y)\big)$, and the restriction of $\kappa_{x,y}$ to $K_{x,y;0+}$ is irreducible by Axiom 4.3.1(4), we can write

$$I^{x,y}_y \circ \Theta_{y|x} \circ (I^{x,y}_x)^{-1} = \Theta' \otimes \mathrm{id}_{V_{\kappa_{x,y}}}$$

for some

$$\Theta' \in \mathrm{Hom}_{K^0_{x,y}}\big(\mathrm{ind}_{K^0_x}^{K^0_{x,y}}(\rho^0_x), \mathrm{ind}_{K^0_y}^{K^0_{x,y}}(\rho^0_y)\big).$$

It remains to show that there exists $c \in \mathcal{C}^\times$ such that $\Theta' = c \cdot \Theta^0_{y|x}$. The definition of $\Theta^0_{y|x}$ implies that

$$\Theta^0_{y|x}\left(V_{\rho^0_x}\right) \subset \mathrm{ind}_{K^0_y}^{K^0_y \cdot K^0_x}\left(V_{\rho^0_y}\right). \qquad (4.3.6a)$$

On the other hand, we can prove

$$\Theta'\left(V_{\rho^0_x}\right) \subset \mathrm{ind}_{K^0_y}^{K^0_y \cdot K^0_x}\left(V_{\rho^0_y}\right) \qquad (4.3.6b)$$

as follows. Let $v^0 \in V_{\rho^0_x}$ and $w \in V_{\kappa_x}$, and consider $v^0 \otimes w \in V_{\rho^0_x} \otimes V_{\kappa_x} = V_{\rho_x} \subset \mathrm{ind}_{K_x}^{K_{x,y}}(V_{\rho_x})$. By tracing through the definition of $I^{x,y}_x$, we obtain that

$$I^{x,y}_x(v^0 \otimes w) = v^0 \otimes w \in V_{\rho^0_x} \otimes V_{\kappa_x} \subset \mathrm{ind}_{K^0_x}^{K^0_{x,y}}(V_{\rho^0_x}) \otimes V_{\kappa_{x,y}},$$

where we regard κ_x as a K_x-subrepresentation of $\kappa_{x,y}$ via $\kappa_x \hookrightarrow \mathrm{ind}_{K_x}^{K^0_x \cdot K_{x,y;0+}}(\kappa_x)|_{K_x} \simeq \kappa_{x,y}|_{K_x}$. Hence, we obtain that $I^{x,y}_x\left(V_{\rho_x}\right) = V_{\rho^0_x} \otimes V_{\kappa_x}$, equivalently, $(I^{x,y}_x)^{-1}\left(V_{\rho^0_x} \otimes V_{\kappa_x}\right) = V_{\rho_x}$. The definitions of $I^{x,y}_y$ and $\Theta_{y|x}$, and the observation $K^0_y \cdot K^0_x \cdot K_{x,y;0+} = K^0_y \cdot K_{x,y;0+} \cdot K^0_x \cdot K_{x,y;0+} \supseteq K_y \cdot K_x$ then imply that

$$\left(I^{x,y}_y \circ \Theta_{y|x} \circ (I^{x,y}_x)^{-1}\right)\left(V_{\rho^0_x} \otimes V_{\kappa_x}\right) \subset \mathrm{ind}_{K^0_y}^{K^0_y \cdot K^0_x}\left(V_{\rho^0_y}\right) \otimes V_{\kappa_{x,y}}.$$

Thus, we conclude (4.3.6b).

According to (4.3.6a) and (4.3.6b), we have

$$\Theta^0_{y|x}|_{V_{\rho^0_x}},\ \Theta'|_{V_{\rho^0_x}} \in \mathrm{Hom}_{K^0_x}\left(\rho^0_x, \mathrm{ind}_{K^0_y}^{K^0_y \cdot K^0_x}(\rho^0_y)\right) \simeq \mathrm{Hom}_{K^0_x}\left(\rho^0_x, \mathrm{ind}_{K^0_x \cap K^0_y}^{K^0_x}(\rho^0_y|_{K^0_x \cap K^0_y})\right),$$

where the isomorphism comes from the isomorphism $\mathrm{ind}_{K_y^0}^{K_y^0\cdot K_x^0}(\rho_y^0) \to \mathrm{ind}_{K_x^0\cap K_y^0}^{K_x^0}(\rho_y^0|_{K_x^0\cap K_y^0})$ of K_x^0-representations given by $f \mapsto f|_{K_x^0}$. Since K_x^0 is a compact group, the compact induction functor $\mathrm{ind}_{K_x^0\cap K_y^0}^{K_x^0}$ is not only the left-adjoint but also the right-adjoint of the restriction functor. Hence, according to Lemma 3.3.5, we have

$$\mathrm{Hom}_{K_x^0}\left(\rho_x^0, \mathrm{ind}_{K_x^0\cap K_y^0}^{K_x^0}(\rho_y^0|_{K_x^0\cap K_y^0})\right) \simeq \mathrm{Hom}_{K_x^0\cap K_y^0}\left(\rho_x^0|_{K_x^0\cap K_y^0}, \rho_y^0|_{K_x^0\cap K_y^0}\right) = \mathrm{End}_{K_{M^0}}(\rho_{M^0}).$$

Since ρ_{M^0} is an irreducible representation of K_{M^0}, we obtain that

$$\dim_{\mathcal{C}}\left(\mathrm{Hom}_{K_x^0}(\rho_x^0, \mathrm{ind}_{K_y^0}^{K_y^0\cdot K_x^0}(\rho_y^0))\right) = \dim_{\mathcal{C}}\left(\mathrm{End}_{K_{M^0}}(\rho_{M^0})\right) = 1.$$

Thus, there exists $c \in \mathcal{C}$ such that $\Theta'|_{V_{\rho_x^0}} = c \cdot \Theta_{y|x}^0|_{V_{\rho_x^0}}$. Since the $K_{x,y}^0$-representation $\big(\mathrm{ind}_{K_x^0}^{K_{x,y}^0}(\rho_x^0), \mathrm{ind}_{K_x^0}^{K_{x,y}^0}(V_{\rho_x^0})\big)$ is generated by the subspace $V_{\rho_x^0}$, and the homomorphisms Θ' and $\Theta_{y|x}^0$ are $K_{x,y}^0$-equivariant, we also obtain that $\Theta' = c\cdot\Theta_{y|x}^0$. Since $\Theta_{y|x}^0$ and $\Theta_{y|x}$ are non-zero (see Corollary 3.5.4), we obtain the lemma. □

Corollary 4.3.7. *We assume Axioms 3.4.1, 4.1.2, 4.2.1, and 4.3.1. Then an affine hyperplane $H \in \mathfrak{H}$ is $\mathcal{K}^0$-relevant if and only if H is $\mathcal{K}$-relevant.*

Proof. Let $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$. To prove the corollary, in light of Equations (4.3.4) and (4.3.5), it suffices to show that

$$\Theta_{x|y} \circ \Theta_{y|x} \in \mathcal{C}\cdot \mathrm{id}_{\mathrm{ind}_{K_x}^{K_{x,y}}(\rho_x)} \qquad \text{if and only if} \qquad \Theta_{x|y}^0 \circ \Theta_{y|x}^0 \in \mathcal{C}\cdot \mathrm{id}_{\mathrm{ind}_{K_x^0}^{K_{x,y}^0}(\rho_x^0)}.$$

According to Lemma 4.3.6, there exists $c \in \mathcal{C}^\times$ such that

$$I_y^{x,y} \circ \Theta_{y|x} \circ (I_x^{x,y})^{-1} = c\cdot \Theta_{y|x}^0 \otimes \mathrm{id}_{V_{\kappa_{x,y}}}.$$

Replacing x with y, we also obtain that there exists $c' \in \mathcal{C}^\times$ such that

$$I_x^{x,y} \circ \Theta_{x|y} \circ \left(I_y^{x,y}\right)^{-1} = c'\cdot \Theta_{x|y}^0 \otimes \mathrm{id}_{V_{\kappa_{x,y}}}.$$

Hence, we have $I_x^{x,y} \circ \left(\Theta_{x|y} \circ \Theta_{y|x}\right) \circ (I_x^{x,y})^{-1} = cc'\cdot \Theta_{x|y}^0 \circ \Theta_{y|x}^0 \otimes \mathrm{id}_{V_{\kappa_{x,y}}}$, from which we deduce the desired equivalence. □

4.4 Hecke algebra isomorphism

We keep the notation and assumptions, i.e., Axioms 3.4.1, 4.1.2, 4.2.1, and 4.3.1, from the previous subsection. In this subsection, we are going to show that the Hecke algebras attached to $(K_{x_0}^0, \rho_{x_0}^0)$ and to (K_{x_0}, ρ_{x_0}), respectively, are isomorphic, see Theorem 4.4.8.

In order to use the structure of the Hecke algebras as a semi-direct product of an affine Hecke algebra with a twisted group algebra that we exhibited in Theorem 3.10.10, we will also need to assume the axioms that were used in that theorem. More precisely, from now on, we suppose that the family $\mathcal{K}^0$ also satisfies Axiom 3.4.3. Starting from after Lemma 4.4.1 we also assume that the group $W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ satisfies Axiom 3.7.1 for a normal subgroup $W(\rho_{M^0})_{\text{aff}}$ of $W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$, and that the family $\mathcal{K}^0$ satisfies Axiom 3.8.2 with the groups $K'_{x,s} = K^0_{x,sx}$ for $s \in S_{\mathcal{K}^0\text{-rel}}$ and $x \in \mathcal{A}_{\text{gen}}$ with $\mathfrak{H}_{x,sx} = \{H_s\}$, and where $K^0_{x,sx}$ denotes the group in Axiom 4.3.1. We will first show that the analogous axioms also hold for $\mathcal{K}$.

Lemma 4.4.1. *The family $\mathcal{K}$ with the subgroup $N(\rho_M)^{\heartsuit}_{[x_0]_M} = N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of $N(\rho_M)_{[x_0]_M}$ satisfies Axiom 3.4.3.*

Proof. Let $x \in \mathcal{A}_{\text{gen}}$. Since the group $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ and the family $\mathcal{K}^0$ satisfy Axiom 3.4.3, we have

$$I_{G^0(F)}(\rho^0_x) = K^0_x \cdot N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cdot K^0_x.$$

Combining this with Axiom 4.2.1(3), we have

$$I_{G(F)}(\rho_x) = K_x \cdot I_{G^0(F)}(\rho^0_x) \cdot K_x = K_x \cdot N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cdot K_x. \qquad \square$$

To show that $\mathcal{K}$ also satisfies the remaining axioms, recall that we chose the group $N(\rho_M)^{\heartsuit}_{[x_0]_M}$ to be $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \subset G^0(F)$. Hence we have

$$N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M = N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K_M = N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K_{M^0} \tag{4.4.2}$$

by Remark 4.1.4. Thus, we obtain that $W(\rho_M)^{\heartsuit}_{[x_0]_M} = W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$. Moreover, according to Corollary 4.3.7, we have $W_{\mathcal{K}\text{-rel}} = W_{\mathcal{K}^0\text{-rel}}$.

Lemma 4.4.3. *The family $\mathcal{K}$ satisfies Axiom 3.8.2 with the group $K'_{x,s} = K_{x,sx}$ for $s \in S_{\mathcal{K}\text{-rel}} = S_{\mathcal{K}^0\text{-rel}}$ and $x \in \mathcal{A}_{\text{gen}}$ with $\mathfrak{H}_{x,sx} = \{H_s\}$, where $K_{x,sx}$ denotes the group in Axiom 4.3.1.*

Proof. According to Remark 4.3.3, the group $K_{x,sx}$ is a compact, open subgroup of $G(F)$ containing K_x. According to Axiom 4.3.1(3), we have

$$\begin{aligned} N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K_{x,sx} &= N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K^0_{x,sx} \cdot K_{x,sx;0+} = N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap G^0(F) \cap K^0_{x,sx} \cdot K_{x,sx;0+} \\ &= N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K^0_{x,sx} \cdot \big(G^0(F) \cap K_{x,sx;0+}\big) = N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K^0_{x,sx}. \end{aligned}$$

Combining this with (4.4.2), we obtain that

$$\big(N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K_{x,sx}\big)/\big(N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K_M\big) = \big(N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K^0_{x,sx}\big)/\big(N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \cap K_{M^0}\big) = \{1, s\},$$

where the last equality follows from $\mathcal{K}^0$ satisfying Axiom 3.8.2 for the group $K'_{x,s} = K^0_{x,sx}$. $\square$

We fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20. According to our assumptions and Proposition 3.10.4, we can choose a family

$$\mathcal{T}^0 = \big\{T^0_n \in \operatorname{Hom}_{K_{M^0}}\big({}^n\rho_{M^0}, \rho_{M^0}\big)\big\}_{n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}}$$

as in Choice 3.10.3.

Proposition 4.4.4. *Assume Axioms 3.4.1, 3.4.3, 3.7.1, 3.8.2, 4.1.2, 4.2.1, and 4.3.1 for the relevant objects as described above. Then there exists a unique family*

$$\mathcal{T} = \{T_n \in \operatorname{Hom}_{K_M}({}^n\rho_M, \rho_M)\}_{n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}}$$

that satisfies all the properties in Choice 3.10.3 and the condition that

$$T_n = T_n^0 \otimes \widetilde{\kappa}_M(n)$$

for all $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ whose projections to $W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ are contained in $\Omega(\rho_{M^0})$.

Proof. According to our assumptions, Lemma 4.4.1, Lemma 4.4.3, and Proposition 3.10.4, we can choose a family $\mathcal{T}$ to satisfy all the properties in Choice 3.10.3. Moreover, according to Remark 3.10.5, there exists a unique family $\mathcal{T}$ satisfying all the properties in Choice 3.10.3 together with the condition that $T_n = T_n^0 \otimes \widetilde{\kappa}_M(n)$ for a set of representatives of lifts $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of $t \in \Omega(\rho_{M^0}) \smallsetminus \{1\}$. Since $\widetilde{\kappa}_M$ is an extension of κ_M and we have $\rho_M \simeq \inf(\rho_{M^0}) \otimes \kappa_M$, we also obtain that $T_n = T_n^0 \otimes \widetilde{\kappa}_M(n)$ for all lifts $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of elements of $\Omega(\rho_{M^0})$. □

We fix a family $\mathcal{T}^0$ as in Choice 3.10.3 and let $\mathcal{T}$ denote the family satisfying the conditions in Proposition 4.4.4. We define non-zero elements $\Phi^0_{x,w} \in \operatorname{End}_{G^0(F)}\left(\operatorname{ind}_{K_x^0}^{G^0(F)}(\rho_x^0)\right)$ and $\Phi_{x,w} \in \operatorname{End}_{G(F)}\left(\operatorname{ind}_{K_x}^{G(F)}(\rho_x)\right)$ for $x \in \mathcal{A}_{\text{gen}}$ and $w \in W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ as in Definition 3.5.23, and let $\varphi^0_{x,w}$, resp., $\varphi_{x,w}$, denote the element of $\mathcal{H}(G^0(F), \rho_x^0)$, resp., $\mathcal{H}(G(F), \rho_x)$, that corresponds to $\Phi^0_{x,w}$, resp., $\Phi_{x,w}$, via the isomorphism in (2.2.3). We write $\varphi^0_w := \varphi^0_{x_0,w}$, $\varphi_w := \varphi_{x_0,w}$, $\Phi^0_w := \Phi^0_{x_0,w}$, and $\Phi_w := \Phi_{x_0,w}$.

According to Theorem 3.10.10, the map

$$\varphi^0_{tw} \mapsto b_t \cdot \mathbb{T}_w \qquad (t \in \Omega(\rho_{M^0}), w \in W(\rho_{M^0})_{\text{aff}})$$

defines an isomorphism of $\mathcal{C}$-algebras

$$\mathcal{I}(\rho^0_{x_0})\colon \mathcal{H}(G^0(F), \rho^0_{x_0}) \xrightarrow{\sim} \mathcal{C}[\Omega(\rho_{M^0}), \mu^{\mathcal{T}^0}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_{M^0})_{\text{aff}}, q^0),$$

and the map

$$\varphi_{tw} \mapsto b_t \cdot \mathbb{T}_w \, (t \in \Omega(\rho_{M^0}), w \in W(\rho_{M^0})_{\text{aff}})$$

defines an isomorphism of $\mathcal{C}$-algebras

$$\mathcal{I}(\rho_{x_0})\colon \mathcal{H}(G(F), \rho_{x_0}) \xrightarrow{\sim} \mathcal{C}[\Omega(\rho_{M^0}), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_{M^0})_{\text{aff}}, q),$$

where $\mu^{\mathcal{T}^0}$ and $\mu^{\mathcal{T}}$ denote the restrictions to $\Omega(\rho_{M^0}) \times \Omega(\rho_{M^0})$ of the 2-cocycles introduced in Notation 3.6.1, and q^0 and q denote the parameter functions $s \mapsto q^0_s$ and $s \mapsto q_s$ from $S_{\mathcal{K}^0\text{-rel}}$ to $\mathcal{C}_{>1}$ such that the elements Φ^0_s and Φ_s satisfy the quadratic relations

$$\left(\Phi^0_s\right)^2 = (q^0_s - 1) \cdot \Phi^0_s + q^0_s \cdot \Phi^0_1 \quad \text{and} \quad (\Phi_s)^2 = (q_s - 1) \cdot \Phi_s + q_s \cdot \Phi_1. \tag{4.4.5}$$

Thus, in order to prove that the two Hecke algebras $\mathcal{H}(G^0(F), \rho^0_{x_0})$ and $\mathcal{H}(G(F), \rho_{x_0})$ are isomorphic, we will prove that $\mu^{\mathcal{T}^0} = \mu^{\mathcal{T}}$ and $q^0 = q$.

Lemma 4.4.6. *We have $\mu^{\mathcal{T}^0} = \mu^{\mathcal{T}}$ on $\Omega(\rho_{M^0}) \times \Omega(\rho_{M^0})$.*

Proof. Let $v, w \in \Omega(\rho_{M^0})$. We fix lifts m of v and n of w in $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$. Since $\widetilde{\kappa}_M$ is a representation, we have $\widetilde{\kappa}_M(mn) = \widetilde{\kappa}_M(m) \circ \widetilde{\kappa}_M(n)$. Hence, our choice of $\mathcal{T}$ implies that

$$\begin{aligned}\mu^{\mathcal{T}}(v,w) \cdot T_{mn} &= T_m \circ T_n = \left(T_m^0 \circ T_n^0\right) \otimes \left(\widetilde{\kappa}_M(m) \circ \widetilde{\kappa}_M(n)\right) \\ &= \left(T_m^0 \circ T_n^0\right) \otimes \widetilde{\kappa}_M(mn) = \mu^{\mathcal{T}^0}(v,w) \cdot T_{mn}^0 \otimes \widetilde{\kappa}_M(mn) = \mu^{\mathcal{T}^0}(v,w) \cdot T_{mn}. \qquad \square\end{aligned}$$

Lemma 4.4.7. *We have $q_s^0 = q_s$ for all $s \in S_{\mathcal{K}^0\text{-rel}}$.*

Proof. Let $x \in \mathcal{A}_{x_0}$ such that $\mathfrak{H}_{x,sx} = \{H_s\}$. Since $\mathfrak{H}_{\mathcal{K}\text{-rel};x_0,sx_0} = \{H_s\}$, by Proposition 3.8.21 and Equation (4.4.5) we have

$$\left(\Phi_{x,s}^0\right)^2 = (q_s^0 - 1) \cdot \Phi_{x,s}^0 + q_s^0 \cdot \Phi_{x,1}^0 \quad \text{and} \quad (\Phi_{x,s})^2 = (q_s - 1) \cdot \Phi_{x,s} + q_s \cdot \Phi_{x,1}. \tag{4.4.7a}$$

Hence, using Proposition 3.8.21, the lemma follows once we show that there exists $c \in \mathcal{C}^\times$ such that

$$(c \cdot \Phi_{x,s})^2 = (q_s^0 - 1) \cdot (c \cdot \Phi_{x,s}) + q_s^0 \cdot \Phi_{x,1}. \tag{4.4.7b}$$

Recall that we have an isomorphism

$$I_x^{x,sx} \colon \operatorname{ind}_{K_x}^{K_{x,sx}}(\rho_x) \xrightarrow{\sim} \inf\left(\operatorname{ind}_{K_x^0}^{K_{x,sx}^0}(\rho_x^0)\right) \otimes \kappa_{x,sx}.$$

Since the representation $\inf\left(\operatorname{ind}_{K_x^0}^{K_{x,sx}^0}(\rho_x^0)\right)$ is trivial on $K_{x,sx;0+}$, and the restriction of the representation $\kappa_{x,sx}$ to $K_{x,sx;0+}$ is irreducible, for any $\Phi \in \operatorname{End}_{K_{x,sx}}\left(\inf\left(\operatorname{ind}_{K_x^0}^{K_{x,sx}^0}(\rho_x^0)\right) \otimes \kappa_{x,sx}\right)$, there exists a unique $\Phi' \in \operatorname{End}_{K_{x,sx}^0}\left(\operatorname{ind}_{K_x^0}^{K_{x,sx}^0}(\rho_x^0)\right)$ such that $\Phi = \Phi' \otimes \mathrm{id}_{V_{\kappa_{x,sx}}}$. Thus, we have an isomorphism of $\mathcal{C}$-algebras

$$\eta_x^{x,sx} \colon \operatorname{End}_{K_{x,sx}^0}\left(\operatorname{ind}_{K_x^0}^{K_{x,sx}^0}(\rho_x^0)\right) \xrightarrow{\sim} \operatorname{End}_{K_{x,sx}}\left(\operatorname{ind}_{K_x}^{K_{x,sx}}(\rho_x)\right)$$

defined by

$$\eta_x^{x,sx}\left(\Phi'\right) = (I_x^{x,sx})^{-1} \circ \left(\Phi' \otimes \mathrm{id}_{V_{\kappa_{x,sx}}}\right) \circ I_x^{x,sx}.$$

Since we assume that the family $\mathcal{K}^0$ satisfies Axiom 3.8.2 with the group $K_{x,sx}^0$, we have a lift of s in $K_{x,sx}^0$, which we also denote by s. Combining this with Axiom 4.3.1(1) and Remark 4.3.3, we have $K_x^0 s K_x^0 \subset K_{x,sx}^0$ and $K_x s K_x \subset K_{x,sx}$. Hence, according to Lemma 2.2.5 and Lemma 3.5.26, we have

$$\Phi_{x,s}^0 \in \operatorname{End}_{G^0(F)}\left(\operatorname{ind}_{K_x^0}^{G^0(F)}(\rho_x^0)\right)_{K_{x,sx}^0} \simeq \operatorname{End}_{K_{x,sx}^0}\left(\operatorname{ind}_{K_x^0}^{K_{x,sx}^0}(\rho_x^0)\right)$$

and

$$\Phi_{x,s} \in \operatorname{End}_{G(F)}\left(\operatorname{ind}_{K_x}^{G(F)}(\rho_x)\right)_{K_{x,sx}} \simeq \operatorname{End}_{K_{x,sx}}\left(\operatorname{ind}_{K_x}^{K_{x,sx}}(\rho_x)\right).$$

We regard $\Phi_{x,s}^0$ and $\Phi_{x,s}$ as elements of the latter spaces and claim that there exists $c \in \mathcal{C}^\times$ such that

$$\eta_x^{x,sx}\left(\Phi_{x,s}^0\right) = c \cdot \Phi_{x,s}.$$

Since $\eta_x^{x,sx}$ is an algebra isomorphism, the first equation of (4.4.7a) would then imply Equation (4.4.7b) as desired.

It remains to prove the claim. The proof is essentially the same as the proof of Lemma 4.3.6. According to Proposition 3.4.18 and Lemma 3.5.26, it suffices to show that $\operatorname{supp}\left((\eta_x^{x,sx})^{-1}(\Phi_{x,s})\right) \subset K_x^0 s K_x^0$. As explained in the proof of Lemma 4.3.6, we have $(I_x^{x,sx})^{-1}\left(V_{\rho_x^0} \otimes V_{\kappa_x}\right) = V_{\rho_x}$, where we regard κ_x as a K_x-subrepresentation of $\kappa_{x,sx}$ via $\kappa_x \to \operatorname{ind}_{K_x}^{K_x^0 \cdot K_{x,sx;0+}}(\kappa_x)|_{K_x} \simeq \kappa_{x,sx}|_{K_x}$. Moreover, since $\operatorname{supp}(\Phi_{x,s}) = K_x s K_x$, Lemma 2.2.5 implies that

$$\left(\Phi_{x,s} \circ (I_x^{x,sx})^{-1}\right)\left(V_{\rho_x^0} \otimes V_{\kappa_x}\right) = \Phi_{x,s}(V_{\rho_x}) \subset \operatorname{ind}_{K_x}^{K_x s K_x}(V_{\rho_x}). \tag{4.4.7c}$$

Since we have $s \in K_{x,sx}^0$ and $K_x^0 \subset K_{x,sx}^0$, we obtain from Axiom 4.3.1(2) that the element s and the group K_x^0 normalize the group $K_{x,sx;0+}$, and hence

$$K_x s K_x \subseteq K_x^0 K_{x,sx;0+} s K_x^0 K_{x,sx;0+} = K_x^0 s K_x^0 K_{x,sx;0+}$$

by Remark 4.3.3. Using the definition of $I_x^{x,sx}$, this allows us to deduce $I_x^{x,sx}\left(\operatorname{ind}_{K_x}^{K_x s K_x}(V_{\rho_x})\right) \subset \operatorname{ind}_{K_x^0}^{K_x^0 s K_x^0}\left(V_{\rho_x^0}\right) \otimes V_{\kappa_{x,y}}$. Precomposing with (4.4.7c), we have $\left(I_x^{x,sx} \circ \Phi_{x,s} \circ (I_x^{x,sx})^{-1}\right)\left(V_{\rho_x^0} \otimes V_{\kappa_x}\right) \subset \operatorname{ind}_{K_x^0}^{K_x^0 s K_x^0}\left(V_{\rho_x^0}\right) \otimes V_{\kappa_{x,y}}$. Thus, we obtain that $\left((\eta_x^{x,sx})^{-1}(\Phi_{x,s})\right)\left(V_{\rho_x^0}\right) \subset \operatorname{ind}_{K_x^0}^{K_x^0 s K_x^0}\left(V_{\rho_x^0}\right)$. Using Lemma 2.2.5 we conclude that $\operatorname{supp}\left((\eta_x^{x,sx})^{-1}(\Phi_{x,s})\right) \subset K_x^0 s K_x^0$. □

Theorem 4.4.8. *We assume the following:*

(1) The family $\mathcal{K}^0$ satisfies Axioms 3.4.1 and 3.4.3 with the subgroup $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of $N(\rho_{M^0})_{[x_0]_{M^0}}$.

(2) The family $\mathcal{K}$ satisfies Axiom 3.4.1 with the subgroup $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of $N(\rho_M)_{[x_0]_M}$.

(3) The group $W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ satisfies Axiom 3.7.1 with a normal subgroup $W(\rho_{M^0})_{\mathrm{aff}}$ of $W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$.

(4) The family $\mathcal{K}^0$ satisfies Axiom 3.8.2 with the group $K'_{x,s} = K^0_{x,sx}$ for each $s \in S_{\mathcal{K}^0\text{-rel}}$ and $x \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,sx} = \{H_s\}$.

We also assume Axioms 4.1.2, 4.2.1, and 4.3.1. We fix a subset $\mathcal{C}_{>1} \subset \mathcal{C}^\times \smallsetminus \{1\}$ as in Choice 3.8.20, choose a family $\mathcal{T}^0$ as in Choice 3.10.3 and let $\mathcal{T}$ denote the family satisfying the conditions in Proposition 4.4.4. Then the map

$$\mathcal{I} \coloneqq (\mathcal{I}(\rho_{x_0}))^{-1} \circ \mathcal{I}(\rho_{x_0}^0)\colon \mathcal{H}(G^0(F), \rho_{x_0}^0) \longrightarrow \mathcal{H}(G(F), \rho_{x_0})$$

is a support-preserving algebra isomorphism.

Proof of Theorem 4.4.8. Combine Theorem 3.10.10 with Lemma 4.4.6 and Lemma 4.4.7. □

We will provide a more explicit description of $\mathcal{I}$ in Theorem 4.4.11 below. In order to do so, we keep the set-up from Theorem 4.4.8 and describe $\mathcal{T}$ more explicitly.

Lemma 4.4.9. *There exists a unique quadratic character $\epsilon : N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \to \{\pm 1\}$ that factors through $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \twoheadrightarrow W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ and is trivial on $\Omega(\rho_{M^0})$ such that*

$$T_n = \epsilon(n) \cdot T_n^0 \otimes \widetilde{\kappa}_M(n) \quad \text{for every} \quad n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}.$$

Proof. Let $w \in W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ and choose a lift $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of w. Since T_n and $T^0_n \otimes \widetilde{\kappa}_M(n)$ are non-zero elements in the one-dimensional space $\mathrm{Hom}_{K_M}({}^n\rho_M, \rho_M)$, there exists $\epsilon(n) \in \mathcal{C}^\times$ such that $T_n = \epsilon(n) \cdot T^0_n \otimes \widetilde{\kappa}_M(n)$. We note that since $\widetilde{\kappa}_M$ is an extension of κ_M and we have $\rho_M \simeq \inf(\rho_{M^0}) \otimes \kappa_M$, our choices of $\mathcal{T}^0$ and $\mathcal{T}$ imply that the scalar $\epsilon(n)$ does not depend on the choice of lift n, but only on w. We write $\epsilon(w) = \epsilon(n)$ for any lift $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of $w \in W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$. Since we chose the family $\mathcal{T}$ to satisfy the condition in Proposition 4.4.4, we have $\epsilon(t) = 1$ for all $t \in \Omega(\rho_{M^0})$. We will prove that $\epsilon(w) \in \{\pm 1\}$ for all $w \in W(\rho_{M^0})_{\mathrm{aff}}$. First, let $s \in S_{\mathcal{K}^0\text{-rel}}$. We fix a lift n_s in $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of s and note that n_s^{-1} is also a lift of $s = s^{-1}$. Using the definitions of $\mu^{\mathcal{T}^0}$ and $\mu^{\mathcal{T}}$, we obtain

$$\begin{aligned}\mu^{\mathcal{T}}(s,s) \cdot T_1 = T_{n_s} \circ T_{n_s^{-1}} &= \epsilon(s)^2 \cdot \left(T^0_{n_s} \circ T^0_{n_s^{-1}}\right) \otimes \left(\widetilde{\kappa}_M(n_s) \circ \widetilde{\kappa}_M(n_s^{-1})\right) \\ &= \epsilon(s)^2 \mu^{\mathcal{T}^0}(s,s) \cdot T^0_1 \otimes \widetilde{\kappa}_M(1) = \epsilon(s)^2 \mu^{\mathcal{T}^0}(s,s) \cdot T_1.\end{aligned}$$

According to Lemma 3.8.10 and Lemma 4.4.7, we have $\mu^{\mathcal{T}^0}(s,s) = q^0_s = q_s = \mu^{\mathcal{T}}(s,s)$. Thus, we conclude that $\epsilon(s)^2 = 1$, that is, $\epsilon(s) \in \{\pm 1\}$, as desired. Next, we consider a general $w \in W(\rho_{M^0})_{\mathrm{aff}}$. We fix a reduced expression $w = s_1 s_2 \cdots s_r$ for w. We fix lifts of s_i for $1 \leq i \leq r$ in $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ and denote them by n_i. We write $n = n_1 n_2 \cdots n_r \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$. Then according to Proposition 3.10.7, we have $T^0_n = T^0_{n_1} \circ T^0_{n_2} \circ \cdots \circ T^0_{n_r}$ and $T_n = T_{n_1} \circ T_{n_2} \circ \cdots \circ T_{n_r}$. Hence, we obtain that $\epsilon(w) = \prod_{i=1}^r \epsilon(s_i) \in \{\pm 1\}$. Moreover, since $W(\rho_{M^0})_{\mathrm{aff}}$ is a Coxeter group, the arguments above imply that the map $w \mapsto \epsilon(w)$ defines a group homomorphism $W(\rho_{M^0})_{\mathrm{aff}} \to \{\pm 1\}$. To prove the lemma, it suffices to show that the character $\epsilon\colon W(\rho_{M^0})_{\mathrm{aff}} \to \{\pm 1\}$ is invariant under the conjugation action by $\Omega(\rho_{M^0})$. Let $t \in \Omega(\rho_{M^0})$ and $w \in W(\rho_{M^0})_{\mathrm{aff}}$ with lifts $\widetilde{t}$ and n in $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$, respectively. Then according to Proposition 3.10.7, we have $T^0_{\widetilde{t}n\widetilde{t}^{-1}} T^0_{\widetilde{t}} = T^0_{\widetilde{t}n} = T^0_{\widetilde{t}} \circ T^0_n$, hence $T^0_{\widetilde{t}n\widetilde{t}^{-1}} = T^0_{\widetilde{t}} \circ T^0_n \circ (T^0_{\widetilde{t}})^{-1}$, and $T_{\widetilde{t}n\widetilde{t}^{-1}} = T_{\widetilde{t}} \circ T_n \circ T^{-1}_{\widetilde{t}}$. Thus, the definition of ϵ implies that $\epsilon(twt^{-1}) = \epsilon(w)$, as required. □

Remark 4.4.10. The quadratic character ϵ from Lemma 4.4.9 extends uniquely to a quadratic character of $\widetilde{K}_M$ that is trivial on K_M. Hence $\epsilon \cdot \widetilde{\kappa}_M$ is a smooth representation of $\widetilde{K}_M$ that extends κ_M. Thus if we had chosen $\epsilon \cdot \widetilde{\kappa}_M$ as the lift $\widetilde{\kappa}_M$ of κ_M in Axiom 4.1.2, then we would obtain $T_n = T^0_n \otimes \widetilde{\kappa}_M(n)$ for every $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$.

The above allows us to describe the isomorphism of Theorem 4.4.8 more explicitly. Note that, as explained in Remark 4.4.10, if one chooses $\widetilde{\kappa}_M$ appropriately, then $\epsilon = 1$.

Theorem 4.4.11. *We assume the same set-up as in Theorem 4.4.8. Then there exists a unique quadratic character* $\epsilon : N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \to \{\pm 1\}$ *that factors through* $N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}} \twoheadrightarrow W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ *and is trivial on* $\Omega(\rho_{M^0})$ *such that the isomorphism* $\mathcal{I}$ *is given as follows. If* $\varphi \in \mathcal{H}(G^0(F), \rho^0_{x_0})$ *is supported on* $K^0_{x_0} n K^0_{x_0}$ *for some* $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$, *then* $\mathcal{I}(\varphi)$ *is supported on* $K_{x_0} n K_{x_0}$ *and satisfies*

$$\mathcal{I}(\varphi)(n) = d_n \cdot \varphi(n) \otimes \left(\epsilon(n) \cdot \widetilde{\kappa}_M(n)\right) \quad \textit{with} \quad d_n = \left(\frac{\left|K^0_{x_0} / \left(K^0_{nx_0} \cap K^0_{x_0}\right)\right|}{\left|K_{x_0} / \left(K_{nx_0} \cap K_{x_0}\right)\right|}\right)^{1/2} \in \mathcal{C}^\times.$$

Proof. If $\varphi \in \mathcal{H}(G^0(F), \rho^0_{x_0})$ is supported on $K^0_{x_0} n K^0_{x_0}$ for some $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$, then φ is a scalar multiple of $\varphi^0_{x_0,w} = \varphi^0_w$ where w denotes the image of n in $W(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$. By the definition of the $\mathcal{C}$-linear isomorphism $\mathcal{I}$, we have $\mathcal{I}(\varphi^0_w) = \varphi_w$. Hence the claim follows using Lemma 3.5.26 and Lemma 4.4.9. □

4.5 Preserving the anti-involutions

In this subsection, we keep the notation from the previous subsection, and we also assume that $\mathcal{C}$ admits a nontrivial involution $c \mapsto \bar{c}$, and that all of the assumptions of Theorem 4.4.8 hold. We fix a family $\mathcal{T}^0$ as in Choice 3.11.5 and let $\mathcal{T}$ denote the family satisfying the conditions in Proposition 4.4.4. We say that a representation (π, V) of a group H is unitary if there exists an H-invariant, positive-definite Hermitian form $\langle\ ,\ \rangle_\pi$ on V.

Lemma 4.5.1. *Suppose that the representation $\widetilde{\kappa}_M$ of $\widetilde{K}_M$ is unitary. Then the family $\mathcal{T}$ satisfies all the properties in Choice 3.11.5.*

Proof. Let $t \in \Omega(\rho_M)$. We will prove that $\varphi^*_t = \varphi_{t^{-1}}$. Since $\varphi^*_t, \varphi_{t^{-1}} \in \mathcal{H}(G(F), \rho_{x_0})_{t^{-1}}$, it suffices to show that $\varphi^*_t(n^{-1}) = \varphi_{t^{-1}}(n^{-1})$ for a lift $n \in N(\rho_{M^0})^{\heartsuit}_{[x_0]_{M^0}}$ of t. According to Equation (3.11.1), it suffices to show that $\langle \varphi_{t^{-1}}(n^{-1}) \cdot v, w\rangle_{\rho_{x_0}} = \langle v, \varphi_t(n) \cdot w\rangle_{\rho_{x_0}}$ for all $v, w \in V_{\rho_{x_0}}$. We may suppose that v and w are of the form $v = v_0 \otimes v_\kappa$ and $w = w_0 \otimes w_\kappa$ for $v_0, w_0 \in V_{\rho^0_{x_0}}$ and $v_\kappa, w_\kappa \in V_{\kappa_{x_0}}$. We fix a $K^0_{x_0}$-invariant, positive-definite Hermitian form $\langle\ ,\ \rangle_{\rho^0_{x_0}}$ on $V_{\rho^0_{x_0}} = V_{\rho_{M^0}}$ and a $\widetilde{K}_M$-invariant, positive-definite Hermitian form $\langle\ ,\ \rangle_{\widetilde{\kappa}_M}$ on $V_{\widetilde{\kappa}_M} = V_{\kappa_M}$. Then the Hermitian form $\langle\ ,\ \rangle_{\rho^0_{x_0}} \otimes \langle\ ,\ \rangle_{\widetilde{\kappa}_M}$ is a K_M-invariant form on $V_{\rho_M} = V_{\rho_{M^0}} \otimes V_{\kappa_M}$. Moreover, since (K_{x_0}, ρ_{x_0}) is a quasi-G-cover of (K_M, ρ_M), the Hermitian form $\langle\ ,\ \rangle_{\rho^0_{x_0}} \otimes \langle\ ,\ \rangle_{\widetilde{\kappa}_M}$ is also a K_{x_0}-invariant form on $V_{\rho_{x_0}} = V_{\rho_M}$. Hence, we may suppose that $\langle\ ,\ \rangle_{\rho_{x_0}} = \langle\ ,\ \rangle_{\rho^0_{x_0}} \otimes \langle\ ,\ \rangle_{\widetilde{\kappa}_M}$. According to Theorem 4.4.11, we have

$$\varphi_t(n) \cdot w = \left(\frac{\left| K^0_{x_0} / \left(K^0_{tx_0} \cap K^0_{x_0} \right) \right|}{\left| K_{x_0} / \left(K_{tx_0} \cap K_{x_0} \right) \right|} \right)^{1/2} \varphi^0_t(n) \cdot w_0 \otimes \widetilde{\kappa}_M(n) \cdot w_\kappa$$

and

$$\varphi_{t^{-1}}(n^{-1}) \cdot v = \left(\frac{\left| K^0_{x_0} / \left(K^0_{t^{-1}x_0} \cap K^0_{x_0} \right) \right|}{\left| K_{x_0} / \left(K_{t^{-1}x_0} \cap K_{x_0} \right) \right|} \right)^{1/2} \varphi^0_{t^{-1}}(n^{-1}) \cdot v_0 \otimes \widetilde{\kappa}_M(n^{-1}) \cdot v_\kappa.$$

Hence, we have

$$\langle \varphi_{t^{-1}}(n^{-1}) \cdot v, w\rangle_{\rho_{x_0}} = \left(\frac{\left| K^0_{x_0} / \left(K^0_{t^{-1}x_0} \cap K^0_{x_0} \right) \right|}{\left| K_{x_0} / \left(K_{t^{-1}x_0} \cap K_{x_0} \right) \right|} \right)^{1/2} \langle \varphi^0_{t^{-1}}(n^{-1}) \cdot v_0, w_0\rangle_{\rho^0_{x_0}} \cdot \langle \widetilde{\kappa}_M(n^{-1}) \cdot v_\kappa, w_\kappa\rangle_{\widetilde{\kappa}_M} \tag{4.5.1a}$$

and

$$\langle v, \varphi_t(n) \cdot w\rangle_{\rho_{x_0}} = \left(\frac{\left| K^0_{x_0} / \left(K^0_{tx_0} \cap K^0_{x_0} \right) \right|}{\left| K_{x_0} / \left(K_{tx_0} \cap K_{x_0} \right) \right|} \right)^{1/2} \langle v_0, \varphi^0_t(n) \cdot w_0\rangle_{\rho^0_{x_0}} \cdot \langle v_\kappa, \widetilde{\kappa}_M(n) \cdot w_\kappa\rangle_{\widetilde{\kappa}_M}. \tag{4.5.1b}$$

Since $t \in \Omega(\rho_M)$, we have $d_{\mathcal{K}\text{-rel}}(x_0, t^{-1}x_0) = \ell_{\mathcal{K}\text{-rel}}(t) = 0$. Hence, according to Corollary 3.5.16, we have $\left|K_{x_0}/\left(K_{t^{-1}x_0} \cap K_{x_0}\right)\right| = \left|K_{t^{-1}x_0}/\left(K_{t^{-1}x_0} \cap K_{x_0}\right)\right|$. Combining this with Axiom 3.4.1(3), we obtain that

$$\left|K_{x_0}/\left(K_{t^{-1}x_0} \cap K_{x_0}\right)\right| = \left|K_{t^{-1}x_0}/\left(K_{t^{-1}x_0} \cap K_{x_0}\right)\right| = \left|K_{x_0}/\left(K_{tx_0} \cap K_{x_0}\right)\right|. \tag{4.5.1c}$$

Similarly, we can prove that

$$\left|K^0_{x_0}/\left(K^0_{t^{-1}x_0} \cap K^0_{x_0}\right)\right| = \left|K^0_{x_0}/\left(K^0_{tx_0} \cap K^0_{x_0}\right)\right|. \tag{4.5.1d}$$

Since the family $\mathcal{T}^0$ satisfies all the properties in Choice 3.11.5, we have

$$\langle \varphi^0_{t^{-1}}(n^{-1}) \cdot v_0, w_0 \rangle_{\rho^0_{x_0}} = \langle (\varphi^0_t)^*(n^{-1}) \cdot v_0, w_0 \rangle_{\rho^0_{x_0}} = \langle v_0, \varphi^0_t(n) \cdot w_0 \rangle_{\rho^0_{x_0}}. \tag{4.5.1e}$$

Moreover, since the Hermitian form $\langle\ ,\ \rangle_{\widetilde{\kappa}_M}$ on $V_{\widetilde{\kappa}_M} = V_{\kappa_M}$ is $\widetilde{K}_M$-invariant, we have

$$\langle \widetilde{\kappa}_M(n^{-1}) \cdot v_\kappa, w_\kappa \rangle_{\widetilde{\kappa}_M} = \langle v_\kappa, \widetilde{\kappa}_M(n) \cdot w_\kappa \rangle_{\widetilde{\kappa}_M}. \tag{4.5.1f}$$

Combining (4.5.1c), (4.5.1d), (4.5.1e), and (4.5.1f) with (4.5.1a) and (4.5.1b), we obtain that $\langle \varphi_{t^{-1}}(n^{-1}) \cdot v, w \rangle_{\rho_{x_0}} = \langle v, \varphi_t(n) \cdot w \rangle_{\rho_{x_0}}$, as desired. □

Corollary 4.5.2. *We assume the same set-up as in Theorem 4.4.8. We also suppose that there exists a $\widetilde{K}_M$-invariant, positive-definite Hermitian form $\langle\ ,\ \rangle_{\widetilde{\kappa}_M}$ on $V_{\widetilde{\kappa}_M} = V_{\kappa_M}$. Then the isomorphism*

$$\mathcal{I}\colon \mathcal{H}(G^0(F), \rho^0_{x_0}) \longrightarrow \mathcal{H}(G(F), \rho_{x_0})$$

in Theorem 4.4.8 preserves the anti-involutions on both sides defined in Section 3.11.

Proof. The corollary follows from Proposition 3.11.7 and Lemma 4.5.1. □

4.6 Application: Equivalence of Bernstein blocks

Throughout this subsection, we suppose that $\mathcal{C} = \mathbb{C}$. We only do so because the literature on types currently makes this assumption.

Let $\mathrm{Rep}(G(F))$ denote the category of smooth $\mathbb{C}$-representations of $G(F)$. The Bernstein decomposition (see [Ber84]) expresses this category as a direct product of full subcategories:

$$\mathrm{Rep}(G(F)) = \prod_{\mathfrak{s} \in \mathfrak{I}(G)} \mathrm{Rep}^{\mathfrak{s}}(G(F)).$$

Here $\mathfrak{I}(G)$ is the set of *inertial equivalence classes*, i.e., equivalence classes $[L, \sigma]_G$ of cuspidal pairs (L, σ) in G, where L is a Levi subgroup of G, σ is an irreducible supercuspidal representation of $L(F)$, and where the equivalence is given by conjugation by $G(F)$ and twisting by unramified characters of $L(F)$. If $\mathfrak{s} = [L, \sigma]_G$, then the block $\mathrm{Rep}^{\mathfrak{s}}(G(F))$ consists of those representations (π, V) for which each irreducible subquotient of π appears as a subquotient in a parabolic induction of $\sigma \otimes \chi$ for some unramified character χ of $L(F)$.

Let K denote a compact, open subgroup of $G(F)$, and (ρ, V_ρ) an irreducible smooth $\mathbb{C}$-representation of K. Let $\mathrm{Rep}_\rho(G(F))$ denote the full subcategory of $\mathrm{Rep}(G(F))$ whose objects are the $\mathbb{C}$-representations (π, V) of $G(F)$ generated by their ρ-isotypic subspaces. Following [BK98, (2.8) and Theorem 4.3(ii)], we have a functor

$$\mathbf{M}_\rho\colon \mathrm{Rep}_\rho(G(F)) \longrightarrow \mathrm{Mod}{-}\mathcal{H}(G(F), \rho) \quad \text{given by} \quad \pi \longmapsto \mathrm{Hom}_{G(F)}(\mathrm{ind}_K^{G(F)}(\rho), \pi),$$

where $\mathrm{Mod}{-}\mathcal{H}(G(F), \rho)$ denotes the category of unital right modules over $\mathcal{H}(G(F), \rho)$. Here, the right action of $\mathcal{H}(G(F), \rho)$ on $\mathrm{Hom}_{G(F)}(\mathrm{ind}_K^{G(F)}(\rho), \pi)$ is given by precomposition using the isomorphism $\mathcal{H}(G(F), \rho) \xrightarrow{\sim} \mathrm{End}_{G(F)}\big(\mathrm{ind}_K^{G(F)}(\rho)\big)$ in (2.2.3).

For $\mathfrak{s} \in \mathfrak{I}(G)$, the pair (K, ρ) is called an $\mathfrak{s}$-*type* if $\mathrm{Rep}_\rho(G(F)) = \mathrm{Rep}^{\mathfrak{s}}(G(F))$. More generally, if $\mathfrak{S}$ is a finite subset of $\mathfrak{I}(G)$, then the pair (K, ρ) is called an $\mathfrak{S}$-*type* if

$$\mathrm{Rep}_\rho(G(F)) = \mathrm{Rep}^{\mathfrak{S}}(G(F)) \coloneqq \prod_{\mathfrak{s} \in \mathfrak{S}} \mathrm{Rep}^{\mathfrak{s}}(G(F)).$$

In either case, by [BK98, (4.3) Theorem (ii)], the functor $\mathbf{M}_\rho$ gives an equivalence of categories

$$\mathrm{Rep}^{\mathfrak{S}}(G(F)) = \mathrm{Rep}_\rho(G(F)) \xrightarrow{\sim} \mathrm{Mod}{-}\mathcal{H}(G(F), \rho). \tag{4.6.1}$$

We write

$$\mathcal{I}^*\colon \mathrm{Mod}{-}\mathcal{H}(G(F), \rho_{x_0}) \xrightarrow{\sim} \mathrm{Mod}{-}\mathcal{H}(G^0(F), \rho^0_{x_0}) \tag{4.6.2}$$

for the equivalence of categories associated to the isomorphism $\mathcal{I}\colon \mathcal{H}(G^0(F), \rho^0_{x_0}) \xrightarrow{\sim} \mathcal{H}(G(F), \rho_{x_0})$ of Theorem 4.4.8.

Theorem 4.6.3. *Let $\mathfrak{S} \subset \mathfrak{I}(G)$ and $\mathfrak{S}_0 \subset \mathfrak{I}(G^0)$ be finite subsets. We suppose that the pairs (K_{x_0}, ρ_{x_0}) and $(K^0_{x_0}, \rho^0_{x_0})$ are an $\mathfrak{S}$-type and an $\mathfrak{S}_0$-type, respectively. Then the functor $(\mathbf{M}_{\rho^0_{x_0}})^{-1} \circ \mathcal{I}^* \circ \mathbf{M}_{\rho_{x_0}}$ gives an equivalence of categories*

$$\mathrm{Rep}^{\mathfrak{S}}(G(F)) \xrightarrow{\sim} \mathrm{Rep}^{\mathfrak{S}_0}(G^0(F)).$$

Proof. This follows from combining the equivalences in 4.6.1 and 4.6.2. □

Theorem 4.6.4. *We choose Haar measures ν and ν^0 on $G(F)$ and $G^0(F)$, and let $\hat{\nu}$ and $\hat{\nu}^0$ denote the corresponding Plancherel measures on the Borel spaces $\mathrm{Irr}_t^{\mathfrak{S}}(G(F))$ and $\mathrm{Irr}_t^{\mathfrak{S}_0}(G^0(F))$ of irreducible, tempered representations in $\mathrm{Rep}^{\mathfrak{S}}(G)$ and $\mathrm{Rep}^{\mathfrak{S}_0}(G^0)$. We suppose that there exists a $\widetilde{K}_M$-invariant, positive-definite Hermitian form $\langle\ ,\ \rangle_{\widetilde{\kappa}_M}$ on $V_{\widetilde{\kappa}_M} = V_{\kappa_M}$. Then the isomorphism $\mathcal{I}\colon \mathcal{H}(G^0(F), \rho^0_{x_0}) \longrightarrow \mathcal{H}(G(F), \rho_{x_0})$ of Theorem 4.4.8 induces a homeomorphism*

$$\mathcal{J}\colon \mathrm{Irr}_t^{\mathfrak{S}_0}(G^0(F)) \longrightarrow \mathrm{Irr}_t^{\mathfrak{S}}(G(F)), \ \textit{such that} \quad \hat{\nu} \circ \mathcal{J} = \dim \kappa_{x_0} \frac{\nu^0(K^0_{x_0})}{\nu(K_{x_0})} \hat{\nu}^0.$$

Proof. The support-preserving isomorphism of Hecke algebras from Theorem 4.4.8 preserves the anti-involutions by Corollary 4.5.2 and hence by [BHK11, 5.2 Proposition] is an isomorphism of Hilbert algebras as defined in [BHK11, §4.1]. From [BHK11, §5.1], one has a homeomorphism $\mathcal{J}\colon \mathrm{Irr}_t^{\mathfrak{S}_0}(G^0(F)) \longrightarrow \mathrm{Irr}_t^{\mathfrak{S}}(G(F))$, such that $\frac{\nu(K_{x_0})}{\dim \rho_{x_0}} \hat{\nu} \circ \mathcal{J} = \frac{\nu^0(K^0_{x_0})}{\dim \rho^0_{x_0}} \hat{\nu}^0$. Our result then follows from the fact that $\dim \rho_{x_0} / \dim \rho^0_{x_0} = \dim \kappa_{x_0}$. □

5 Hecke algebras of depth-zero pairs

In this section, we will show that all of the axioms of Section 3 apply to the special case of a pair (K_{x_0}, ρ_{x_0}) where K_{x_0} is a normal, compact, open subgroup of $G(F)_{[x_0]}$ that contains the parahoric subgroup $G(F)_{x_0,0}$, and the restriction of ρ_{x_0} to $G(F)_{x_0,0}$ contains the inflation of a cuspidal representation of $\mathsf{G}_{x_0}(\mathfrak{f}) = G(F)_{x_0,0}/G(F)_{x_0,0+}$. Thus, Theorem 3.10.10 applies to such pairs. This means that we obtain an explicit description of the corresponding Hecke algebras as a semi-direct product of an affine Hecke algebra with a twisted group algebra, see Theorem 5.3.6. The special case where $K_{x_0} = G(F)_{x_0,0}$ and $\mathcal{C} = \mathbb{C}$ has already been treated by Morris [Mor93, Theorem 7.12]. The resulting types describe certain finite products of Bernstein blocks. Our construction also includes the case $K_{x_0} = G(F)_{x_0}$, when the resulting types describe single Bernstein blocks.

5.1 Construction of depth-zero pairs

We recall the notion of a depth-zero G-datum following Kim and Yu ([KY17]), but adjusted to our more general coefficient field $\mathcal{C}$, from which the pair (K_{x_0}, ρ_{x_0}) will be constructed.

Definition 5.1.1 (cf. [KY17, 7.1])**.** A *depth-zero G-datum* is a triple $\big((G, M), (x_0, \iota), (K_M, \rho_M)\big)$ such that

(1) M is a Levi subgroup of G.

(2) $x_0 \in \mathcal{B}(M, F)$ is a point whose image under the projection to $\mathcal{B}^{\text{red}}(M, F)$ is a vertex, and $\iota\colon \mathcal{B}(M, F) \hookrightarrow \mathcal{B}(G, F)$ is a 0-generic admissible embedding relative to x_0 in the sense of [KY17, Definition 3.2], i.e., $M(F)_{x_0,0}/M(F)_{x_0,0+} \simeq G(F)_{\iota(x_0),0}/G(F)_{\iota(x_0),0+}$. We use this embedding to identify $\mathcal{B}(M, F)$ with its image in $\mathcal{B}(G, F)$.

(3) K_M is a compact, open subgroup of $M(F)_{x_0}$ containing $M(F)_{x_0,0}$, and ρ_M is an irreducible smooth $\mathcal{C}$-representation of K_M such that $\rho_M|_{M(F)_{x_0,0}}$ is the inflation of a cuspidal representation of $\mathsf{M}_{x_0}(\mathfrak{f}) = M(F)_{x_0,0}/M(F)_{x_0,0+}$.

If $\mathcal{C} = \mathbb{C}$, a depth-zero G-datum is used to construct types for all depth-zero Bernstein blocks based on works of Moy and Prasad ([MP96]) and Bushnell and Kutzko ([BK98]), which is a special case of the construction below.

From now on, we let $\Sigma = \big((G, M), (x_0, \iota), (K_M, \rho_M)\big)$ be a depth-zero G-datum. We note that for each $x \in \mathcal{A}_{x_0} \coloneqq x_0 + (X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R})$ such that $\iota : \mathcal{B}(M, F) \to \mathcal{B}(G, F)$ is 0-generic relative to x, the triple $\Sigma_x = \big((G, M), (x, \iota), (K_M, \rho_M)\big)$ is also a G-datum. For $x \in \mathcal{A}_{x_0}$, we set

$$K_x = K_M \cdot G(F)_{x,0} \quad \text{and} \quad K_{x,+} = G(F)_{x,0+}, \tag{5.1.2}$$

which are compact, open subgroups of $G(F)$. We note that if $\iota : \mathcal{B}(M, F) \hookrightarrow \mathcal{B}(G, F)$ is 0-generic relative to x, then we have $K_x = K_M \cdot K_{x,+}$ since K_M contains $M_{x_0,0} = M_{x,0}$. In this case, by [KY17, 4.3 Proposition (b)], the inclusion $K_M \subseteq K_x$ induces an isomorphism

$$K_M/M(F)_{x,0+} \xrightarrow{\sim} K_x/K_{x,+},$$

and we define the irreducible smooth representation ρ_x of $K_x/K_{x,+}$ as the composition of ρ_M with the inverse of the isomorphism above. We also regard ρ_x as an irreducible smooth representation of K_x that is trivial on $K_{x,+}$. If $\mathcal{C} = \mathbb{C}$, the pair (K_x, ρ_x) is a type.

If $N_G(M)(F)_{[x_0]_M}$ normalizes the group K_M, e.g., if $K_M = M(F)_{x_0}$ or $M(F)_{x_0,0}$, then we will show in Section 5.3 that the objects G, M, x_0, K_M, ρ_M and the families $\{(K_x, K_{x,+}, \rho_x)\}$ (for appropriate $x \in \mathcal{A}_{x_0}$) satisfy all the desired axioms of Section 3 for the choice of $N(\rho_M)^{\heartsuit}_{[x_0]_M} = N(\rho_M)_{[x_0]_M}$.

5.2 Affine hyperplanes

In order to apply Section 3 to the objects introduced in Section 5.1, we also need an appropriate set of affine hyperplanes as in Section 3.2. We define these hyperplanes as follows. For a maximal split torus S of G, let $\Phi(G,S)$ denote the relative root system of G with respect to S and let $\Phi_{\mathrm{aff}}(G,S)$ denote the (relative) affine root system associated to (G,S) by the work of Bruhat and Tits [BT72]. We fix a maximal split torus S of M such that $x_0 \in \mathcal{A}(G,S,F)$. For $a \in \Phi_{\mathrm{aff}}(G,S) \smallsetminus \Phi_{\mathrm{aff}}(M,S)$, we define the affine hyperplane H_a in $\mathcal{A}(G,S,F)$ by

$$H_a = \{x \in \mathcal{A}(G,S,F) \mid a(x) = 0\}.$$

Since $a \notin \Phi_{\mathrm{aff}}(M,S)$, the intersection $\mathcal{A}_{x_0} \cap H_a$ is an affine hyperplane in $\mathcal{A}_{x_0}$. We define the locally finite set $\mathfrak{H}_S$ of affine hyperplanes in $\mathcal{A}_{x_0}$ by

$$\mathfrak{H}_S = \{\mathcal{A}_{x_0} \cap H_a \mid a \in \Phi_{\mathrm{aff}}(G,S) \smallsetminus \Phi_{\mathrm{aff}}(M,S)\}.$$

Lemma 5.2.1. *The set of affine functionals $\left\{a|_{\mathcal{A}_{x_0}} \mid a \in \Phi_{\mathrm{aff}}(G,S) \smallsetminus \Phi_{\mathrm{aff}}(M,S)\right\}$ on $\mathcal{A}_{x_0}$ and the set $\mathfrak{H}_S$ do not depend on the choice of a maximal split torus S of M.*

Proof. Let S' be another maximal split torus of M such that $x_0 \in \mathcal{A}(G,S',F)$. Then there exists an element $m \in M(F)_{x_0,0}$ such that $mSm^{-1} = S'$, and we obtain a bijection between $\Phi_{\mathrm{aff}}(G,S) \smallsetminus \Phi_{\mathrm{aff}}(M,S)$ and $\Phi_{\mathrm{aff}}(G,S') \smallsetminus \Phi_{\mathrm{aff}}(M,S')$ by sending $a \in \Phi_{\mathrm{aff}}(G,S) \smallsetminus \Phi_{\mathrm{aff}}(M,S)$ to the affine root $ma \in \Phi_{\mathrm{aff}}(G,S') \smallsetminus \Phi_{\mathrm{aff}}(M,S')$ defined by $(ma)(x) = a(m^{-1}x)$ for $x \in \mathcal{A}(G,S',F) = m\mathcal{A}(G,S,F)$. Since the group $M(F)_{x_0,0}$ acts trivially on $\mathcal{A}_{x_0}$, for every $a \in \Phi_{\mathrm{aff}}(G,S) \smallsetminus \Phi_{\mathrm{aff}}(M,S)$, we have $(ma)(x) = a(x)$ for all $x \in \mathcal{A}_{x_0}$, i.e., $(ma)|_{\mathcal{A}_{x_0}} = a|_{\mathcal{A}_{x_0}}$ and

$$\mathcal{A}_{x_0} \cap H_a = \{x \in \mathcal{A}_{x_0} \mid a(x) = 0\} = \{x \in \mathcal{A}_{x_0} \mid (ma)(x) = 0\} = \mathcal{A}_{x_0} \cap H_{ma}.$$

Thus, we obtain that $\mathfrak{H}_S = \mathfrak{H}_{S'}$. ☐

Based on the lemma, we can set $\mathfrak{H} = \mathfrak{H}_S$, where S is any maximal split torus of M such that $x_0 \in \mathcal{A}(G,S,F)$. Then $\iota : \mathcal{B}(M^0,F) \to \mathcal{B}(G^0,F)$ is 0-generic relative to $x \in \mathcal{A}_{x_0}$ if and only if x is not contained in any affine hyperplane $H \in \mathfrak{H}$, that is, $x \in \mathcal{A}_{\mathrm{gen}} = \mathcal{A}_{x_0} \smallsetminus \left(\bigcup_{H \in \mathfrak{H}} H\right)$. In particular, we have $x_0 \in \mathcal{A}_{\mathrm{gen}}$.

Lemma 5.2.2. *The action of $N_G(M)(F)_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ preserves the set $\mathfrak{H}$.*

Proof. Let $n \in N_G(M)(F)_{[x_0]_M}$. We fix a maximal split torus S of M such that $x_0 \in \mathcal{A}(G,S,F)$. Then the torus nSn^{-1} is also a maximal split torus of M, and we have

$$x_0 \in nx_0 + (X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}) \subseteq n\mathcal{A}(G,S,F) = \mathcal{A}(G, nSn^{-1}, F).$$

Since the set $\mathfrak{H}_S$ does not depend on the choice of such a maximal split torus of M, we obtain that $n(\mathfrak{H}_S) = \mathfrak{H}_{nSn^{-1}} = \mathfrak{H}_S$. Thus, we conclude that $n(\mathfrak{H}) = \mathfrak{H}$. ☐

5.3 The structure of Hecke algebras attached to depth-zero pairs

From now on, we suppose that the group K_M is normalized by $N_G(M)(F)_{[x_0]_M}$. For instance, if we choose $K_M = M(F)_{x_0}$ or $M(F)_{x_0,0}$, then this assumption is satisfied. We impose this assumption to show that the support of the Hecke algebra attached to (K_x, ρ_x) is given by $K_x \cdot N(\rho_M)_{[x_0]_M} \cdot K_x$, see Proposition 5.3.2 below. If $\mathcal{C} = \mathbb{C}$, then the case of $K_M = M(F)_{x_0}$ corresponds to types for single Bernstein blocks, while the case $K_M = M(F)_{x_0,0}$ is the one that Morris ([Mor93]) studied.

In this subsection, we will prove Theorem 5.3.6, i.e., that the Hecke algebra $\mathcal{H}(G(F), \rho_{x_0})$ is isomorphic to a semi-direct product of an affine Hecke algebra with a twisted group algebra, by verifying all the required axioms from Section 3 that allow us to apply Theorem 3.10.10. We recall that we have constructed in Section 5.1 the family

$$\mathcal{K} = \{(K_x, K_{x,+}, (\rho_x, V_{\rho_x}))\}_{x \in \mathcal{A}_{\text{gen}}}$$

of quasi-G-cover-candidates as defined at the beginning of Section 3.4. We will now prove that the family also satisfies Axioms 3.4.1 and 3.4.3 for the group $N(\rho_M)^{\heartsuit}_{[x_0]_M} := N(\rho_M)_{[x_0]_M}$.

Lemma 5.3.1.

(1) For every $x \in \mathcal{A}_{\text{gen}}$, we have

(a) $K_{nx} = nK_x n^{-1}$ and $K_{nx,+} = nK_{x,+}n^{-1}$ for $n \in N(\rho_{M^0})_{[x_0]_{M^0}}$,

(b) the pair (K_x, ρ_x) is a quasi-G-cover of (K_M, ρ_M),

(c) $K_x = K_M \cdot K_{x,+}$,

(d) $K_{x,+} = (K_{x,+} \cap U(F)) \cdot (K_{x,+} \cap M(F)) \cdot \big(K_{x,+} \cap \overline{U}(F)\big)$ for all $U \in \mathcal{U}(M)$.

Moreover, the group $K_{x,+} \cap M(F)$ is independent of the point $x \in \mathcal{A}_{\text{gen}}$.

(2) For $x, y, z \in \mathcal{A}_{\text{gen}}$ such that $d(x, y) + d(y, z) = d(x, z)$, there exists $U \in \mathcal{U}(M)$ such that

$$K_x \cap U(F) \subseteq K_y \cap U(F) \subseteq K_z \cap U(F) \quad \text{and} \quad K_z \cap \overline{U}(F) \subseteq K_y \cap \overline{U}(F) \subseteq K_x \cap \overline{U}(F).$$

Thus, the family $\mathcal{K}$ satisfies Axiom 3.4.1 for the group $N(\rho_M)^{\heartsuit}_{[x_0]_M} = N(\rho_M)_{[x_0]_M}$.

Proof. The first claim follows from the definitions and [KY17, 4.3 Proposition, Theorem 7.5]. We will prove the second claim. Let $x, y, z \in \mathcal{A}_{\text{gen}}$ such that $d(x, y) + d(y, z) = d(x, z)$. Recall that $z - x$ is an element of $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. We take $U \in \mathcal{U}(M)$ such that $\alpha(z - x) \geq 0$ for any non-zero weight $\alpha \in X^*(A_M)$ occurring in the adjoint representation of A_M on the Lie algebra of U. Then the definition of the groups K_x and K_z implies that

$$\begin{cases} K_x \cap U(F) & \subseteq K_z \cap U(F), \\ K_z \cap \overline{U}(F) & \subseteq K_x \cap \overline{U}(F). \end{cases} \tag{5.3.1a}$$

We will prove that

$$K_x \cap U(F) \subseteq K_y \cap U(F) \subseteq K_z \cap U(F).$$

Suppose that $K_x \cap U(F) \not\subseteq K_y \cap U(F)$. Then the definitions of K_x and K_y imply that there exists $a \in \Phi_{\mathrm{aff}}(G,S)$ such that the gradient $Da \in \Phi(G,S)$ occurs in the adjoint representation of S on the Lie algebra of U, and

$$a(x) > 0 \qquad \text{and} \qquad a(y) < 0.$$

In particular we have $\mathcal{A}_{x_0} \cap H_a \in \mathfrak{H}_{x,y}$. Since $d(x,y)+d(y,z) = d(x,z)$, Lemma 3.2.1 implies that $\mathfrak{H}_{x,y}, \mathfrak{H}_{y,z} \subset \mathfrak{H}_{x,z}$. Hence, we also obtain that $\mathcal{A}_{x_0} \cap H_a \in \mathfrak{H}_{x,z}$. Since $a(x) > 0$, we have $a(z) < 0$. Then the definitions of K_x and K_z imply that $K_x \cap U_{Da}(F) \not\subseteq K_z \cap U_{Da}(F)$, where U_{Da} denotes the root subgroup corresponding to Da. However, this contradicts (5.3.1a). Hence, we obtain that $K_x \cap U(F) \subseteq K_y \cap U(F)$. Similarly, we can prove that $K_y \cap U(F) \subseteq K_z \cap U(F)$. Replacing x with z and U with $\overline{U}$, we also obtain that

$$K_z \cap \overline{U}(F) \subseteq K_y \cap \overline{U}(F) \subseteq K_x \cap \overline{U}(F).$$

This proves the second claim. The first two properties of Axiom 3.4.1 follow from Lemma 5.2.2, the remaining properties from the first two claims proven above. □

Next, we will prove that the family $\mathcal{K}$ satisfies Axiom 3.4.3. The following proposition is essentially a translation of [Mor93, 4.15 Theorem] into our slightly more general setting.

Proposition 5.3.2. *We have*

$$K_x \cdot N(\rho_M)_{[x_0]_M} \cdot K_x = I_{G(F)}(\rho_x)$$

for all $x \in \mathcal{A}_{\mathrm{gen}}$, that is, the family $\mathcal{K}$ satisfies Axiom 3.4.3 for $N(\rho_M)^{\heartsuit}_{[x_0]_M} = N(\rho_M)_{[x_0]_M}$.

To prove Proposition 5.3.2, we prepare the following lemma. For a maximal split torus S of G and $x \in \mathcal{A}(G,S,F)$, we write

$$\Phi_{\mathrm{aff},x}(G,S) = \{a \in \Phi_{\mathrm{aff}}(G,S) \mid a(x) = 0\}.$$

Lemma 5.3.3. *Let $x, y \in \mathcal{A}_{x_0}$. As above, we denote by S a maximal split torus of M such that $x_0 \in \mathcal{A}(G,S,F)$. Suppose that $x \in \mathcal{A}_{\mathrm{gen}}$. Then we have*

$$\Phi_{\mathrm{aff},x}(G,S) \subseteq \Phi_{\mathrm{aff},y}(G,S).$$

Proof. Let $b \in \Phi_{\mathrm{aff},x}(G,S)$. Since $x \notin H_a$ for all $a \in \Phi_{\mathrm{aff}}(G,S) \smallsetminus \Phi_{\mathrm{aff}}(M,S)$, we have $b \in \Phi_{\mathrm{aff}}(M,S)$. Thus, using that $y-x \in X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$, we obtain $b(y) = b(x+(y-x)) = b(x)+Db(y-x) = b(x) = 0$. Hence $\Phi_{\mathrm{aff},x}(G,S) \subseteq \Phi_{\mathrm{aff},y}(G,S)$. □

Proof of Proposition 5.3.2. Let $x \in \mathcal{A}_{\mathrm{gen}}$. It suffices to prove that $K_x \cdot N(\rho_M)_{[x_0]_M} \cdot K_x \supseteq I_{G(F)}(\rho_x)$ because the reverse inclusion follows from Corollary 3.3.6. Hence let $g \in I_{G(F)}(\rho_x)$. We fix a maximal split torus S of M such that $x_0 \in \mathcal{A}(G,S,F)$. According to [KP23, Theorem 7.8.1], we have $G(F) = K_x \cdot N_G(S)(F) \cdot K_x$. Hence, we may suppose that $g \in I_{G(F)}(\rho_x) \cap N_G(S)(F)$. Since $\mathrm{Hom}_{K_x \cap {}^gK_x}({}^g\rho_x, \rho_x) \neq \{0\}$, and the representation ${}^g\rho_x$ is trivial on the group ${}^gG(F)_{x,0+}$, we obtain that ρ_x has a non-zero $(K_x \cap {}^gG(F)_{x,0+})$-fixed vector. In particular, we obtain that the representation ρ_x has a non-zero $(G(F)_{x,0} \cap {}^gG(F)_{x,0+})$-fixed vector. We have that

$$(G(F)_{x,0} \cap {}^gG(F)_{x,0+})\, G(F)_{x,0+}/G(F)_{x,0+} = (G(F)_{x,0} \cap G(F)_{gx,0+})\, G(F)_{x,0+}/G(F)_{x,0+}$$

is the group of $\mathfrak{f}$-points of the unipotent radical of a parabolic subgroup of G_x, and the restriction of ρ_x to $\mathsf{G}_x(\mathfrak{f}) = \mathsf{M}_x(\mathfrak{f})$ is a cuspidal representation, so we obtain that

$$(G(F)_{x,0} \cap G(F)_{gx,0+})\, G(F)_{x,0+}/G(F)_{x,0+} = \{1\}.$$

In particular, we have

$$(M(F)_{x,0} \cap M(F)_{gx,0+})\, M(F)_{x,0+}/M(F)_{x,0+} = \{1\}.$$

Since the point $[x]_M$ is a vertex in $\mathcal{A}^{\mathrm{red}}(M, S, F)$, this equation implies that $[gx]_M = [x]_M$, that is,

$$gx \in x + (X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}) = \mathcal{A}_{x_0}.$$

Since $x \in \mathcal{A}_{\mathrm{gen}}$, Lemma 5.3.3 implies that $\Phi_{\mathrm{aff},x}(G, S) \subset \Phi_{\mathrm{aff},gx}(G, S)$. Since $|\Phi_{\mathrm{aff},x}(G, S)| = |g\Phi_{\mathrm{aff},x}(G, S)| = |\Phi_{\mathrm{aff},gx}(G, S)|$, we obtain that $\Phi_{\mathrm{aff},x}(G, S) = \Phi_{\mathrm{aff},gx}(G, S)$. In particular, we have $gx \notin H_a$ for all $a \in \Phi_{\mathrm{aff}}(G, S) \smallsetminus \Phi_{\mathrm{aff}}(M, S)$. Since $x, gx \notin H_a$ for all $a \in \Phi_{\mathrm{aff}}(G, S) \smallsetminus \Phi_{\mathrm{aff}}(M, S)$, and the projection $[gx]_M = [x]_M$ is a vertex, we have

$$A_M = \Big(\bigcap_{a \in \Phi_{\mathrm{aff},x}(G,S)} \ker(Da) \Big)^{\circ} = \Big(\bigcap_{a \in \Phi_{\mathrm{aff},gx}(G,S)} \ker(Da) \Big)^{\circ} = {}^{g}A_M.$$

Thus, we obtain that $g \in N_G(M)(F)$. Moreover, since $gx \in \mathcal{A}_{x_0}$, Corollary 3.1.3 implies that $g \in N_G(M)(F)_{[x_0]_M}$. As $N_G(M)(F)_{[x_0]_M}$ normalizes K_M, this also implies that g normalizes the group K_M. Combining this with the assumption $g \in I_{G(F)}(\rho_x)$ and Lemma 3.3.5, we obtain that $g \in N_{G(F)}(\rho_M)$. Hence $g \in N_{G(F)}(\rho_M) \cap N_G(M)(F)_{[x_0]_M} = N(\rho_M)_{[x_0]_M}$. □

We recall from Definition 3.4.15 that

$$W(\rho_M)^{\heartsuit}_{[x_0]_M} = N(\rho_M)^{\heartsuit}_{[x_0]_M} / \Big(N(\rho_M)^{\heartsuit}_{[x_0]_M} \cap K_M \Big) = N(\rho_M)_{[x_0]_M}/K_M$$

and from Section 3.7 that $W_{\mathcal{K}\text{-rel}} = \langle s_H \mid H \in \mathfrak{H}_{\mathcal{K}\text{-rel}} \rangle$ with set of simple reflections $S_{\mathcal{K}\text{-rel}}$, see Notation 3.8.1. To prove Axioms 3.7.1 and 3.8.2 we first introduce some notation.

Notation 5.3.4. Let $x, y \in \mathcal{A}_{\mathrm{gen}}$ with $d(x, y) = 1$. We denote by $H_{x,y} \in \mathfrak{H}$ the unique hyperplane that satisfies $\mathfrak{H}_{x,y} = \{H_{x,y}\}$ and define the compact, open subgroup $K_{x,y}$ of $G(F)$ by $K_{x,y} = K_h = K_M \cdot G(F)_{h,0}$, where $h \in H_{x,y}$ is the unique point for which $h = x + t \cdot (y - x)$ for some $0 < t < 1$.

Since $d(x, y) = 1$, the definition of $\mathfrak{H}$ implies that we have $G(F)_{x,0}, G(F)_{y,0} \subseteq G(F)_{h,0}$. Thus, we have $K_x, K_y \subseteq K_{x,y}$.

Proposition 5.3.5.

(1) There exists a normal subgroup $W(\rho_M)_{\mathrm{aff}}$ of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ such that the action of $W(\rho_M)^{\heartsuit}_{[x_0]_M}$ on $\mathcal{A}_{x_0}$ restricts to an isomorphism

$$W(\rho_M)_{\mathrm{aff}} \xrightarrow{\sim} W_{\mathcal{K}\text{-rel}},$$

that is, Axiom 3.7.1 is satisfied. Moreover, the elements of $W(\rho_M)_{\mathrm{aff}}$ can be represented by elements in $G_{\mathrm{cpt},0} \cap N(\rho_M)_{[x_0]_M}$, where $G_{\mathrm{cpt},0}$ denotes the kernel of the Kottwitz homomorphism on $G(F)$.

(2) For every $s \in S_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,sx} = \{H_s\}$, we have

$$\left(N(\rho_M)_{[x_0]_M} \cap K_{x,sx}\right)/K_M = \{1, s\},$$

where $K_{x,sx}$ denotes the group in Notation 5.3.4. Thus, the family $\mathcal{K}$ satisfies Axiom 3.8.2 with $N(\rho_M)^{\heartsuit}_{[x_0]_M} = N(\rho_M)_{[x_0]_M}$ and the group $K'_{x,s} = K_{x,sx}$ for each $s \in S_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,sx} = \{H_s\}$.

Proof. First, we will prove (1). Note that we have $G_{\mathrm{cpt},0} \cap M(F)_{x_0} = G_{\mathrm{cpt},0} \cap M(F)_{x_0,0}$. Hence, according to Lemma 3.7.2, to prove (1), it suffices to show that for all $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, there exists an element

$$s'_H \in \left(G_{\mathrm{cpt},0} \cap N(\rho_M)_{[x_0]_M}\right) / \left(G_{\mathrm{cpt},0} \cap K_M\right)$$

such that the action of s'_H on $\mathcal{A}_{x_0}$ agrees with the orthogonal reflection s_H. Let $H \in \mathfrak{H}_{\mathcal{K}\text{-rel}}$, and $x, y \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,y} = \{H\}$ and

$$\Theta_{x|y} \circ \Theta_{y|x} \notin \mathcal{C} \cdot \mathrm{id}_{\mathrm{ind}_{K_x}^{G(F)}(\rho_x)}. \tag{5.3.5a}$$

Since $K_x, K_y \subset K_{x,y}$, the definitions of $\Theta_{x|y}$ and $\Theta_{y|x}$ imply that

$$\Theta_{x|y} \circ \Theta_{y|x} \in \mathrm{End}_{G(F)}\left(\mathrm{ind}_{K_x}^{G(F)}(\rho_x)\right)_{K_{x,y}} \simeq \mathrm{End}_{K_{x,y}}\left(\mathrm{ind}_{K_x}^{K_{x,y}}(\rho_x)\right),$$

where the isomorphism follows from Lemma 2.2.8. Hence, (5.3.5a) implies that the dimension of the $\mathcal{C}$-vector space

$$\mathrm{End}_{K_{x,y}}\left(\mathrm{ind}_{K_x}^{K_{x,y}}(\rho_x)\right) \simeq \mathcal{H}(K_{x,y}, \rho_x)$$

is greater than one. According to Proposition 3.4.18, a basis of the space $\mathcal{H}(K_{x,y}, \rho_x)$ is indexed by the group

$$\left(N(\rho_M)_{[x_0]_M} \cap K_{x,y}\right)/K_M \subset W(\rho_M)^{\heartsuit}_{[x_0]_M}.$$

In particular, we can take a non-trivial element $s'_H \in \left(N(\rho_M)_{[x_0]_M} \cap K_{x,y}\right)/K_M$. Recall that $K_{x,y} = K_M \cdot G(F)_{h,0}$, where $h \in H$ is the unique point such that $h = x + t \cdot (y - x)$ for some $0 < t < 1$. Hence, we have

$$\begin{aligned} s'_H \in \left(N(\rho_M)_{[x_0]_M} \cap K_{x,y}\right)/K_M &\simeq \left(G_{h,0} \cap N(\rho_M)_{[x_0]_M}\right) / \left(G_{h,0} \cap K_M\right) \\ &\subset \left(G_{\mathrm{cpt},0} \cap N(\rho_M)_{[x_0]_M}\right) / \left(G_{\mathrm{cpt},0} \cap K_M\right), \end{aligned}$$

where the isomorphism is given by the inclusion of $G_{h,0}$ into $K_{x,y}$. We will prove that the action of the element s'_H on the space $\mathcal{A}_{x_0}$ agrees with the orthogonal reflection with respect to the affine hyperplane H. It suffices to show the following three properties.

(i) The gradient of the action of s'_H on $\mathcal{A}_{x_0}$ preserves the inner product $(\ ,\ )_M$ on $X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$.

(ii) The action of s'_H on $\mathcal{A}_{x_0}$ is nontrivial.

(iii) For any $z \in H$, we have $s'_H(z) = z$.

The first property follows from the fact that $(\ ,\ )_M$ is preserved by $N_G(M)(F)$. The second property follows from the facts that $s'_H \neq 1$ and that the group $\left(G_{\mathrm{cpt},0} \cap N(\rho_M)_{[x_0]_M}\right) / \left(G_{\mathrm{cpt},0} \cap K_M\right)$ acts faithfully on $\mathcal{A}_{x_0}$ by Remark 3.4.16 combined with $G_{\mathrm{cpt},0} \cap M(F)_{x_0} = G_{\mathrm{cpt},0} \cap K_M$.

We will prove the third property. It suffices to show that there exists a non-empty open subset U of H such that $s'_H(z) = z$ for all $z \in U$. Since $\mathfrak{H}_{x,y} = \{H\}$, the definitions of $\mathfrak{H}$ and h and the parahoric subgroups imply that there exists an open ball U in H with center h such that $G_{h,0} = G_{z,0}$ for all $z \in U$. Hence, we have $s'_H \in \left(G_{z,0} \cap N(\rho_M)_{[x_0]_M}\right) / \left(G_{z,0} \cap K_M\right)$ for all $z \in U$. In particular, $s'_H(z) = z$ for all $z \in U$.

Next, we will prove (2). Let $s \in S_{\mathcal{K}\text{-rel}}$ and $x \in \mathcal{A}_{\mathrm{gen}}$ such that $\mathfrak{H}_{x,sx} = \{H_s\}$. By Part (1) the element s can be represented by an element in $G_{\mathrm{cpt},0} \cap N(\rho_M)_{[x_0]_M}$, which we also denote by s. Since s fixes H_s pointwise, we have $s \in G_{\mathrm{cpt},0} \cap N(\rho_M)_{[x_0]_M} \cap G_{h'}$ for any $h' \in H_s$. Hence $s \in K_{x,sx}$. Thus $\left(N(\rho_M)_{[x_0]_M} \cap K_{x,sx}\right) / K_M \supset \{1, s\}$. If $s'_H \in \left(N(\rho_M)_{[x_0]_M} \cap K_{x,sx}\right) / K_M$ is nontrivial, then the same arguments as in the proof of Part (1) show that that $s'_H = s$. □

Now we have shown that all the axioms of Section 3 are satisfied in the setting of the present section. Thus, we obtain the following result.

Theorem 5.3.6. *We have an isomorphism of $\mathcal{C}$-algebras*

$$\mathcal{H}(G(F), \rho_{x_0}) \simeq \mathcal{C}[\Omega(\rho_M), \mu^{\mathcal{T}}] \ltimes \mathcal{H}_{\mathcal{C}}(W(\rho_M)_{\mathrm{aff}}, q),$$

where $\mu^{\mathcal{T}}$ denotes the restriction to $\Omega(\rho_M) \times \Omega(\rho_M)$ of the 2-cocycle introduced in Notation 3.6.1 for a choice of a family $\mathcal{T}$ satisfying the properties of Choice 3.10.3 and q denotes the parameter function $s \mapsto q_s$ appearing in Choice 3.10.3(3). If $\mathcal{C}$ admits a nontrivial involution, then we can choose $\mathcal{T}$ as in Choice 3.11.5, and the above isomorphisms can be chosen to preserve the anti-involutions on each algebra defined in Section 3.11.

Proof. The statement follows from Theorem 3.10.10 and Proposition 3.11.7, whose assumptions are satisfied by Lemma 5.3.1, Proposition 5.3.2, and Proposition 5.3.5. □

In the case where K_{x_0} is a parahoric subgroup of $G(F)$, this result was proven by Morris ([Mor93, 7.12 Theorem]).

List of axioms

Selected notation

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