TIME-FREQUENCY ANALYSIS AND CARLESON'S THEOREM: A STUDENT'S POINT OF VIEW

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Abstract. The goal of this article is to study the proof of Carleson's Theorem by M. Lacey and C. Thiele, contained in [8]. Throughout this proccess, the relation between this proof and the recently sharpened methods of time-frequency analysis will be clear, besides the links of the problem with some other classical problems in harmonic analysis.

1. Introduction

Given a function $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$ such that $f \in L^2$, its *Fourier Transform* is defined as the sequence of complex numbers $\{\widehat{f}(k)\}_{k\in\mathbb{Z}}$ given by

$$\widehat{f}(k) = \int_{-1/2}^{1/2} e^{-2\pi ikx} f(x)\mathrm{d}x.$$

One of the most fundamental facts in harmonic analysis is that, for this class of functions, the *Parseval Identity* holds. It guarantees that

$$\sum_{|k|\leq N} \widehat{f}(k)e^{2\pi ixk} \xrightarrow{L^2} f(x).$$

We may ask ourselves, therefore, if the convergence holds almost everywhere (with respect to the Lebesgue measure on [-1/2,1/2]).

The hugely celebrated theorem of Carleson [1], in 1966, gives the affirmative answer to such question. In the subsequente years, a considerably big amount of effort has been made to understand Lennart Carleson's techniques to solve the problem. In 1968, Hunt [5] proved that we may also prove this convergence for $f \in L^p$, $1 < p < 2$. It is worth to notice also that, way before Carleson could prove the veracity of the theorem for square-integrable functions, Kolmogorov had already proved that we cannot obtain the same convergence in the L^1 case , showing even that there are functions whose Fourier series diverge *everywhere*. As proving this is not our main goal, we leave to the interested reader to check his original article ([6]).

In 1973, Charles Fefferman ([2]), in one of the articles that were responsible for his Fields Medal, gave a new, simpler proof of the almost everywhere convergence of Fourier Series in L^2. Effectively, Fefferman used some ideas originally used by Carleson in [1], introducing a *Maximal operator* associated to the problem, which is called, nowadays, the *Carleson operator*, defined by

$$\mathcal{C}f(x) = \sup_{N\in\mathbb{Z}} \left| \sum_{|k|\leq N} \widehat{f}(k) e^{2\pi i x k} \right|,$$

and proving that it satisfies an estimate of the type

$$\|\mathcal{C}f\|_1 \leq C\|f\|_2$$

for every $f \in L^2$. Years later, Michael Lacey and Christoph Thiele ([8]), inspired by the techniques contained in Fefferman and Carleson's work, proved a generalization in the context of Fourier integrals: Given a $f \in L^2(\mathbb{R})$, if we define its *Carleson operator* as

$$\mathcal{C}f(x) = \sup_{N\in\mathbb{R}} \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{-2\pi i x \xi} \mathrm{d}x \right|,$$

then it can be proved that there is a constant $A > 0$ such that for all $f \in L^2$ and all λ,

$$m(\{x \in \mathbb{R}; \mathcal{C}f(x) > \lambda\}) \leq \frac{A}{\lambda^2}\|f\|_2^2. \tag{1}$$

We call this type of inequality a *weak-type (2,2) inequality*. The conection between a weak estimate and pointwise convergence is fundamental in harmonic analysis. We may already conclude the theorem assuming the estimate (1) to be true. In fact, if $\widehat{f} \in C_c^\infty(\mathbb{R})$, then

$$f_R(\xi) = \int_{-\infty}^{R} \widehat{f}(x) e^{-2\pi i x \xi} \mathrm{d}x \xrightarrow{R\to\infty} f(\xi).$$

From the most basic properties of the Schwartz class (see, for example, [3, Section 2.2]), we know that the set of $f \in \mathcal{S}$ whose $\mathrm{supp}\widehat{f}$ is compact is dense in all the L^p spaces. In this way, given any $f \in L^2$ and $\varepsilon, \eta > 0$, pick a $g \in (C_c^\infty)^\vee$ such that $\|f - g\|_2 < \varepsilon\eta$.

$$\begin{aligned}
&m(\{x \in \mathbb{R}; \limsup f_R(x) - \liminf f_R(x) > \varepsilon\}) = \\
&\leq m(\{x \in \mathbb{R}; \limsup (f-g)_R(x) - \liminf (f-g)_R(x) > \varepsilon\}) \\
&\leq \frac{A}{\varepsilon^2}\|f-g\|_2^2 \leq A\eta^2.
\end{aligned}$$

As η was arbitrary, we may conclude that, in fact, the measure of all the sets above is null, which implies that the lim sup coincides with the lim inf almost everywhere. By Plancherel's Theorem (see [3]), we have that this limit must agree with the function almost everywhere, which completes the proof.

So, in order to understand completely Carleson's Theorem, we must study the estimate (1). To do so, we will follow the main ideas on the original article [8], in the auxiliary articles [7], [9] and the general ideas concerning wave packet and Time-Frequency analysis in the book [10] and in [4, Chapter 11]

2. Basic Concepts and Primary Definitions

Before entering the toughest part, we must introduce the basic notation to be used throughout this paper. Therefore, when we write down a function ϕ, we mean an element $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that its Fourier Transform is a positive function $\widehat{\phi} \in C_c^\infty$ such that $\text{supp}\widehat{\phi} \subset [-1/10, 1/10], \widehat{\phi} \equiv 1$ em $[-0.09, 0.09]$. Besides, we will pick a measureable, integrable function w (weight function) defined by

$$w(x) = (1 + |x|)^{-R},$$

where $R \in \mathbb{N}$ is a big natural number, whose specific value is not interesting to us. We define also the *η-Modulation*, *y-Translation* and Λ, p*-dilation* by, respectively,

$$\begin{aligned} M_\eta f(x) &= e^{2\pi i \eta x} f(x); \\ T_y f(x) &= f(x - y); \\ D^p_\Lambda f(x) &= \Lambda^{-1/p} f(\Lambda^{-1} x). \end{aligned}$$

It is immediate to verify that all these operators are L^p isometries. In particular, ncerning these operators when $p = 2$ and their action on L^2, we have the following:

Proposition 1. *$\forall f \in L^2(\mathbb{R}^n)$, we have that*

$$\begin{aligned} \widehat{M_\eta f} &= T_\eta \widehat{f}; \\ \widehat{T_y f} &= M_{-y} \widehat{f}; \\ \widehat{D^2_\Lambda f} &= D^2_{1/\Lambda} \widehat{f}. \end{aligned}$$

Proof. For the Proof, see [3, Proposition 2.2.11]. □

For the proof of our result, we need a fundamental concept that will allow us to reconstruct and linearize the Carleson operator. In this way, we see that an interval I is *dyadic* if if there are two integers $n, k \in \mathbb{Z}$ such that $I = [2^k n, 2^k(n + 1))$.

The set of all dyadic intervals will be denoted by $\mathcal{D}$. Dyadic intervals are central objects in harmonic analysis, but our focus will remain mainly in slightly more complex objects these induce: we say that a dyadic rectangle $s \in \mathcal{D} \times \mathcal{D}$ is a *tile* if $s = I \times \omega$, where $|I||\omega| = 1$. The interval I is called the tile's *space component*, whereas the interval ω is called the *frequential component* of the tile. We will denote the set of all tiles by $\mathcal{T}$. To achieve our mentioned goal of linearizing the Carleson Operator, we need to find an efficient way to localize (in the phase plane) our objects of study.

Given a dyadic interval ω, its *upper part* is defined as the interval $\omega_+ = [c(\omega), +\infty) \cap \omega$, where $c(J)$ is the center of the interval J. We define, then, the *lower part* of such an interval as $\omega_- = \omega \backslash \omega_+$. Given these notions, we build the *adapted bump function* with respect to the tile $s = I \times \omega$ as

$$\begin{aligned}\phi_s &= M_{c(\omega_-)} T_{c(I)} D^2_{|I|}\phi \\ &= e^{2\pi i x c(\omega_-)}|I|^{-1/2}\phi\left(\frac{x-c(I)}{|I|}\right).\end{aligned} \tag{2}$$

Using Proposition 1 and the definition of ϕ, we have that the Fourier Transform $\widehat{\phi_s}$ is supported in $\frac{1}{5}\omega_-$. We also define

$$\begin{aligned} w_s(x) &= w_{I_s}(x) \\ &= T_{c(I_s)} D^1_{|I_s|} w(x) \\ &= \frac{1}{|I_s|} w\left(\frac{x-c(I_s)}{|I_s|}\right).\end{aligned}$$

From these definitions, we see immediately that $|\phi(x)| \le C|I_s|^{1/2}|w_s(x)|$, where C does not depend on s. These objects so defined will play an essential role through the unfolding of the Theorem's proof.

To finish this first part, we remember two classical Theorems from Fourier Analysis that are going to be undicriminatedly used through the text:

Proposition 2 (Necessary Background). *The following items hold:*

(a) *For all $f, g \in L^2(\mathbb{R}^n)$,*

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle.$$

(b) *If an operator $Q : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ commutes with T_y, D^2_Λ, and has in its kernel the functions f whose* $\operatorname{supp}\widehat{f} \subseteq [0, +\infty)$*, then there is $c \in \mathbb{R}$ such that*

$$Qf(x) = c\int_{-\infty}^{0} \widehat{f}(\xi) e^{2\pi i x\xi} \mathrm{d}\xi.$$

Proof. See [3, Chapter 2] for this and many other interesting properties of the Fourier Transform and Fourier Multiplier Operators. □

We will generally refer to Proposition 2, part (a), as *Plancherel's Theorem.* Throughout the text, $C, C', C'', C_1, C_2, C_3$ will always be absolute constants, whose values are not necessarily equal, and may change from line to line.

3. The Core of the Proof

Our primary goal will be to reconstruct the Carleson Operator, employing some basic ideas from harmonic analysis:

(i) The use of the functions $\{\phi_s\}_{s\in\mathcal{T}}$ to work as an 'almost' orthonormal basis, as they allow us to control our function successfully in space and frequency
(ii) A way to Characterize of an Operator by its Group of Symmetries, or simply by its relation with symmetries.
(iii) The Linearization the Study of an Operator by looking at it with duality statements rather than crude measure-theoretical conditions.

3.1. **Decomposition of the Carleson Operator.** Instead of analyzing the Operator $\int_{-\infty}^{\xi} \widehat{f}(t)e^{2\pi i x t}\mathrm{d}t$, we wish to build some sort of 'discrete' analog of it. Put more clearly, we want to come up with an Operator that captures most of the properties of this one, but is easier to handle. Using the idea in (i) above, we may take:

$$Q_\xi f = \sum_{s\in\mathcal{T}} \chi_{\omega_{s+}}(\xi)\langle f, \phi_s\rangle \phi_s.$$

Notice that this has one essential feature that our operators have: if a term in the sum above is nonzero, then $\xi \in \omega_{s+}$, supp $\widehat{\phi_s} \subset \omega_{s-}$. Therefore, supp $\widehat{\phi_s} \subset (-\infty, \xi)$. However, we still do not know if this Operator is really well-defined. That is, among other properties, the content of the next proposition. Before proving it, we define the auxiliary operators

$$Q_\xi^m f = \sum_{s\in\mathcal{T}_m} \chi_{\omega_{s+}}\langle f, \phi_s\rangle \phi_s,$$

where $\mathcal{T}_m = \{s \in \mathcal{T}; |I_s| = 2^{-m}\}$.

Proposition 3. *Let $\xi \in \mathbb{R}$ and $m, k \in \mathbb{Z}$. Then:*

(a) *The operators Q_ξ^m are bounded in L^2, with bound depending neither on m nor on ξ.*
(b) *The operator $Q_\xi = \sum_{m\in\mathbb{Z}} Q_\xi^m$ is bounded in L^2, with bound independent of ξ.*
(c) *The following identity holds:*

$$Q_\xi = D^2_{2^{-k}} Q_{2^{-k}\xi} D^2_{2^k}.$$

Proof. (a) We start by noting that $\langle \phi_s, \phi_{s'}\rangle \chi_{\omega_{s+}}(\xi)\chi_{\omega_{s-}}(\xi) \neq 0 \Rightarrow \omega_s = \omega_{s'}$. Indeed, Plancherel's Theorem gives that, for this sum to be nonzero, it is necessary that supp $\widehat{\phi_s} \cap$ supp $\widehat{\phi_{s'}} \neq \emptyset \Rightarrow \omega_{s-} \cap \omega_{s'-} \neq \emptyset$. On the other hand, the conditions on ξ give that the upper parts of the intervals also intersect. This can only happen if $\omega_s = \omega_{s'}$.

Next,we write, using the intersection property above,

$$\begin{aligned}\left\| \sum_{s\in\mathcal{T}_m} \chi_{\omega_{s+}}(\xi)\langle f, \phi_s\rangle\phi_s \right\|_2^2 &\leq \sum_{s,s'\in\mathcal{T}_m;\omega_s=\omega_{s'}} \chi_{\omega_{s+}}(\xi)\chi_{\omega_{s'+}}(\xi)|\langle \phi_s, \phi_{s'}\rangle\langle f, \phi_s\rangle\langle f, \phi_{s'}\rangle| \\ &\leq \sum_{s,s'\in\mathcal{T}_m;\omega_s=\omega_{s'}} \chi_{\omega_{s+}}(\xi)|\langle f, \phi_s\rangle|^2|\langle \phi_s, \phi_{s'}\rangle| \\ &\leq C_1 \sum_{s,s'\in\mathcal{T}_m;\omega_s=\omega_{s'}} |\langle f, \phi_s\rangle|^2\chi_{\omega_{s+}}(\xi),\end{aligned}$$

where we have used a Cauchy-Schwarz inequality and that, for $s \in \mathcal{T}_m$, then

$$\sum_{s'\in\mathcal{T}_m;\omega_{s'}=\omega_s} |\langle \phi_s, \phi_{s'}\rangle| \leq C_1$$

(You may check, for example, Lemma 6). Now, we start to estimate:

$$
\begin{aligned}
|\langle f, \phi_s \rangle| &\leq C|I_s|^{-1/2} \int_{\mathbb{R}} |f(y)| \left(1 + \frac{|y - c(I_s)|}{|I_s|}\right)^{-N} \mathrm{d}y \\
&\leq C'|I_s|^{1/2} \int_{\mathbb{R}} |f(y)| \left(1 + \frac{|y - z|}{|I_s|}\right)^{-N} \mathrm{d}y \\
&\leq C''|I_s|^{1/2} Mf(z),
\end{aligned}
$$

for all $z \in I_s$, where M denotes the Hardy-Littlewood Maximal operator and we have used [3, Theorem 2.1.10]. This gives us directly

$$|\langle f, \phi_s \rangle|^2 \leq C_2 \int_{I_s} Mf(z)^2 \mathrm{d}z.$$

This, along with the previous considerations and the fact that $M : L^2 \to L^2$ boundedly, gives the desired conclusion for part (a).

(b) We are just going to use part (a) in a nice way. First, note that for $m \in \mathbb{Z}$ fixed, there is exactly one dyadic interval with length 2^{-k} for which $\xi \in \omega_+$. Let ω_m be this interval, and $\omega_{m'} = \omega_{m-}$. Define, then, the functions f_m by $\widehat{f_m} = \chi_{\omega_{m'}} \widehat{f}$. From the properties of the functions ϕ_s and Plancherel's Theorem, we get that $Q_\xi^m(f_m) = Q_\xi^m(f)$. Moreover, if $m \neq n \Rightarrow \omega_{m'} \cap \omega_{n'} = \emptyset$, because $\omega_{m+} \cap \omega_{n+} \neq \emptyset \Rightarrow \omega_m \subset \omega_{n+}$ or vice-versa. Then, from the disjointness property,

$$
\begin{aligned}
\left\| \sum_{m \in \mathbb{Z}} Q_\xi^m f \right\|_2^2 &= \sum_{m \in \mathbb{Z}} \|Q_\xi^m f\|_2^2 \\
&= \sum_{m \in \mathbb{Z}} \|Q_\xi^m (f_m)\|_2^2 \\
&\leq C \sum_{m \in \mathbb{Z}} \|f_m\|_2^2 \\
&= C \sum_{m \in \mathbb{Z}} \|\widehat{f_m}\|_2^2 \\
&= C\|\widehat{f}\|_2^2 = C\|f\|_2^2.
\end{aligned}
$$

This already shows both that the operator Q_ξ is well-defined in L^2 and that it is bounded with bound not depending on ξ, as desired.

(c) This is just a calculation involving the definitions of Q_ξ and of ϕ_s, and because of that we omit this proof. □

3.2. **Reconstruction of the Carleson Operator.** Now we are able to really employ our second idea: find a way to take into account the symmetries the operators of partial Fourier inversion have. This will follow a few steps:

First, let $M_{-\eta} T_{-y} D^2_{2^{-\lambda}} Q^m_{\frac{\xi+\eta}{2^\lambda}} D^2_{2^\lambda} T_y M_\eta = P^m_{\xi,y,\eta,\lambda}$. This expression has the easily verified property that it is *periodic in* y *and* η. In fact, we know exactly the period: in y, it is $2^{m-\lambda}$, and in η, it is $2^{\lambda-m}$. Moreover, the following lemma will help us bound the size of $P^m_{\xi,y,\eta,\lambda} f$.

Lemma 1. *Fix a function $f \in \mathcal{S}(\mathbb{R})$, a tile s and ξ, η, y, λ. Then, for sufficiently large m depending only on ξ,*

$$\chi_{\omega_{s+}}\left(\frac{\xi+\eta}{2^\lambda}\right)|\langle D^2_{2^\lambda} T_y M_\eta f, \phi_s\rangle| \leq C_f \min(1, 2^m).$$

Proof. The '1' is the easiest part: using a Cauchy-Schwarz inequality yields this. Therefore, our focus will be on the other inequality. Using the properties of these operators and Plancherel's theorem, we see that, in the case where $\xi + \eta \in 2^\lambda \omega_{s+}$,

$$\begin{aligned}
|\langle D^2_{2^\lambda} T_y M_\eta f, \phi_s\rangle| &= |\langle f, M_{-\eta} T_{-y} D^2_{2^{-\lambda}} \phi_s\rangle| \\
&= |\langle \widehat{f}, T_{-\eta} M_y D^2_{2^\lambda} \widehat{\phi_s}\rangle| \\
&\leq 2\|\widehat{f}\|_{L^1((-\infty, -\frac{1}{40\cdot 2^m})\cup(\frac{1}{40\cdot 2^m}, +\infty))}\|\widehat{\phi_s}\|_\infty \\
&\leq C2^{m/2}\|\widehat{f}\|_{L^1((-\infty, -\frac{1}{40\cdot 2^m})\cup(\frac{1}{40\cdot 2^m}, +\infty))} \\
&\leq C_f 2^m,
\end{aligned}$$

where we have used that (i) $\widehat{\phi_s}$ has nice support properties; (ii) $\frac{\xi+\eta}{2^\lambda} \in \omega_{s+}$; (iii) Hölder's Inequality; (iv) $f \in \mathcal{S}(\mathbb{R})$. This clearly completes the proof.

□

Using this Lemma and some algebraic manipulation, along with the fact that $\phi \in \mathcal{S}$, we see that $|P^m_{\xi,y,\eta,\lambda} f| \leq C_f \min(2^{-m/2}, 2^{m/2})$, where C_f depends on f but not on ξ, y, η, λ.

We want now to perform averages to catch the symmetries of our partial Fourier inversion operators. More specifically, we want to show that the limit

$$\lim_{N,L\to\infty} \frac{1}{4NL}\int_0^1 \int_{-N}^N \int_{-L}^L P^m_{\xi,y,\eta,\lambda} f \,\mathrm{d}y\,\mathrm{d}\eta\,\mathrm{d}\lambda$$

exists indeed. The next lemma will then play a crucial role:

Lemma 2. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a measurable, bounded and periodic function of period P_1 on the first variable and P_2 on the second one. Then*

$$\lim_{K_1,K_2\to\infty} \frac{1}{4K_1K_2}\int_{-K_1}^{K_1}\int_{-K_2}^{K_2} g(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{P_1P_2}\int_0^{P_1}\int_0^{P_2} g(x,y)\mathrm{d}x\,\mathrm{d}.$$

Proof. Write $K_1 = P_1T_1 + r_1,\ K_2 = P_2T_2 + r_2$, where $T_i \in \mathbb{N}, r_i \in [0, P_i], i = 1, 2.$ Then:

$$\frac{1}{4K_1K_2}\int_{-K_1}^{K_1}\int_{-K_2}^{K_2} g(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{4K_1K_2}\int_{-K_1}^{K_1}\int_{-P_2T_2}^{P_2T_2} g(x,y)\mathrm{d}x\mathrm{d}y + \frac{1}{4K_1K_2}\int_{-K_1}^{K_1}\int_{[-K_2,K_2]\setminus[-P_2T_2,P_2T_2]} g(x,y)\mathrm{d}x\mathrm{d}y$$
$$= \frac{2T_2}{4K_1K_2}\int_{-K_1}^{K_1}\int_{0}^{P_2} g(x,y)\mathrm{d}x\mathrm{d}y + \frac{1}{4K_1K_2}\int_{-K_1}^{K_1}\int_{[-K_2,K_2]\setminus[-P_2T_2,P_2T_2]} g(x,y)\mathrm{d}x\mathrm{d}y$$

But the second term is small, because the function is bounded, and the integral will then be over a set of measure $\leq 2K_1$, still normalized by $4K_1K_2$. The ratio T_2/K_2 is close, while $K_2 \to \infty$, to $1/P_2$. Applying the same procedure to the other coordinate gives us the result.

□

Now, to conclude the existence of the limits above, it suffices to note that, as the bound $|P^m_{\xi,y,\eta,\lambda} f| \leq C_f \min(2^{-m/2}, 2^{m/2})$ tells us, the limit

$$\lim_{N,L\to\infty} \frac{1}{4NL}\int_{-N}^{N}\int_{-L}^{L} P^m_{\xi,y,\eta,\lambda} f \mathrm{d}y\, \mathrm{d}\eta$$

is also bounded by $C_f \min(2^{-m/2}, 2^{m/2})$. Using the Dominated Convergence Theorem, we may change the order of limit with the integral, getting the desired conclusion.

From the bound we have and using the Dominated Convergence Theorem twice, we see that the following limit

$$\begin{aligned}
\Pi_\xi f := &\lim_{N,L\to\infty} \frac{1}{4NL}\int_{-N}^{N}\int_{-L}^{L}\int_0^1 M_{-\eta}T_{-y}D^2_{2^{-\lambda}}Q_{\frac{\xi+\eta}{2^\lambda}}D^2_{2^\lambda}T_y M_\eta f \mathrm{d}y\mathrm{d}\eta\mathrm{d}\lambda \\
= &\lim_{N,L\to\infty}\sum_{m\in\mathbb{Z}} \frac{1}{4NL}\int_0^1\int_{-N}^{N}\int_{-L}^{L} P^m_{\xi,y,\eta,\lambda} f\mathrm{d}y\mathrm{d}\eta\mathrm{d}\lambda \\
= &\sum_{m\in\mathbb{Z}} \lim_{N,L\to\infty} \frac{1}{4NL}\int_0^1\int_{-N}^{N}\int_{-L}^{L} P^m_{\xi,y,\eta,\lambda} f\mathrm{d}y\mathrm{d}\eta\mathrm{d}\lambda \\
= &\sum_{m\in\mathbb{Z}} \Pi^m_\xi f
\end{aligned}$$

also exists. This new operators $\Pi_\xi f$ are our means to rebuild the partial Fourier inversion operators. In fact, our next theorem contains much of what is needed to prove it:

Theorem 1. *Let $\xi \in \mathbb{R}$. Then the following items hold:*

(i) $\|\Pi_\xi f\|_2 \leq C\|f\|_2$ *for a constant $C > 0$ independent from ξ.*

(ii) $M_{-\theta}\Pi_{\xi+\theta}M_\theta = \Pi_\xi$, *for all $\xi, \theta \in \mathbb{R}$.*

(iii) Π_ξ *is a positive semidefinite, nonzero operator.*

(iv) *The Operators* $M_{-\xi}\Pi_\xi M_\xi$ *commute with dilations and translations, besides containing on their kernels the functions* $h \in L^2$ *such that* supp $\widehat{h} \subset [0, +\infty)$.

Proof. (i) This follows directly from Fatou's Lemma and Proposition 3.

(ii) Explicitly, we have that

$$M_{-\theta} \circ P^m_{\xi+\theta,y,\eta,\lambda} \circ M_\theta f =$$

$$= \sum_{s\in\mathcal{T}_m} \chi_{\omega_{s+}}\left(\frac{\xi+\theta+\eta}{2^\lambda}\right) \langle D^2_{2^\lambda} T_y M_{\theta+\eta} f, \phi_s\rangle M_{-\theta-\eta} T_{-y} D^2_{2^{-\lambda}}\phi_s$$

so that when we integrate with respect to the modulating factors, we will have, instead of an integral from $-L$ to L, one from $-L+\theta$ to $L+\theta$. As our procedure is a limiting one, this will disappear as we grow L, and, thus, both limits must be the same.

(iii) That this operator is positive semidefinite comes from examining the expression $\langle \Pi_\xi f, f\rangle$, looking at each Π^m_ξ sepparately, throwing the limits out of the inner products and exchanging orders of integration. In fact, doing all that, we end up with

$$\langle \Pi_\xi f, f\rangle = \lim \frac{1}{4KL}\int_{-L}^{L}\int_{-K}^{K}\int_0^1 \sum_{s\in\mathcal{T}} \chi_{\omega_{s+}}((\xi+\eta)/2^\lambda)|\langle D^2_{2^\lambda} T_y M_\eta f, \phi_s\rangle|^2 \mathrm{d}y\, \mathrm{d}\eta\, \mathrm{d}\lambda.$$

To notice it is nonzero, it suffices to notice that the Lemma 2 along with restricting to $s \in \mathcal{T}_0$ (tiles that are, in fact, unit squares) shows us that it is enough to find a function $h \in \mathcal{S}(\mathbb{R})$ such that, for some tile $s \in \mathcal{T}_0$, $\langle D^2_{2^\lambda} T_y M_\eta h, \phi_s\rangle \neq 0$ for all λ, y, η in a small neighbourhood of zero. This comes from the fact that

$$\langle \Pi_\xi h, h\rangle \geq \int_0^1 \int_0^{2^{-\lambda}} \int_0^{2^\lambda} \sum_{s\in\mathcal{T}_0} \chi_{\omega_{s+}}((\xi+\eta)/2^\lambda)|\langle D^2_{2^\lambda} T_y M_\eta h, \phi_s\rangle|^2 \mathrm{d}y\, \mathrm{d}\eta\, \mathrm{d}\lambda.$$

In order to achieve this, pick first a tile such that $\xi \in \omega_{s+}$ and then a range of numbers η, λ small enough so that $2^{-\lambda}(\xi+\eta) \in \omega_{s+}$ in this range. Take, then, $h = \phi_s$, the function adapted to this tile. For a small enough range contained in the previous one, as dilations, modulations and translations are continuous operators, the inner product wanted is nonzero. In that case, the integral will be strictly positive also, as we wanted.

(iv) The proofs that the operators $M_{-\xi}\Pi_\xi M_\xi$ commute with dilations and translations follows the same ideas of the proof of (ii), except that we have to use that $D^2_b M_\eta = M_{\eta/b} D^2_b$, $T_z M_\eta = e^{-2\pi i\eta z} M_\eta T_z$ and $D^2_b T_z = T_{bz} D^2_b$. Because of the not enlightening at all calculations, we omit them. Finally, the support property follows easily from a direct computation with the property that supp $\widehat{\phi_s} \subset \omega_{s-}$. □

We are ready to prove the reconstruction property: in fact, the Proposition 2, part (b), gives us immediately that, as $M_{-\xi}\Pi_\xi M_\xi$:
(i) commutes with translations and dilations;
(ii) is nonzero;
(iii) is positive semidefinite;
(iv) has on its kernel the functions h such that supp $\widehat{h} \subset [0, +\infty)$;

then it must be the case where $M_{-\xi}\Pi_\xi M_\xi = c_\xi \int_{-\infty}^0 \widehat{f}(t) e^{2\pi i t x} \mathrm{d}t$, for some $c_\xi \in \mathbb{R}\backslash\{0\}$. From the item (ii) of the Theorem 1, we see that $c_\xi = c_{\theta+\xi} = c$ for all $\theta, \xi \in \mathbb{R}$. This also allows us to write

$$\Pi_\xi f(x) = c \int_{-\infty}^{\xi} \widehat{f}(t) e^{2\pi i t x} \mathrm{d}t.$$

Thus,

$$\mathcal{C}f(x) = \frac{1}{|c|} \sup_{\xi \in \mathbb{R}} |\Pi_\xi f|.$$

3.3. **Linearization of the Problem.** In this section, we are going to further reduce our problem to a more treatable one, with the aid of the tools we have built on the first two subsections. This one will consist only of a single proposition:

Proposition 4. *In order to prove that the Carleson Operator is a weak type (2,2) operator, it is enough to check that there is an universal constant* $C > 0$ *such that, for all* $f \in \mathcal{S}(\mathbb{R})$; $\|f\|_2 = 1$, $N : \mathbb{R} \to \mathbb{R}$ *measurable,* $E \subset \mathbb{R}; |E| \in (1, 2]$ *measurable set and* $\boldsymbol{P} \subset \mathcal{T}$ *finite set of tiles, we have*

$$\sum_{s \in \boldsymbol{P}} |\langle f, \phi_s \rangle \langle \phi_s, \chi_{E \cap N^{-1}(\omega_{s+})} \rangle| \leq C. \tag{3}$$

The inequality 3 is is what we are going to call our *Key Estimate.* After we prove the proposition, we will focus entirely on this new estimate that arose.

Proof. Step 1. First, we introduce the quasinorm

$$\|g\|_{2,\infty} = \sup_{\lambda \in \mathbb{R}_+} \lambda m(\{x; |g(x)| > \lambda\})^{1/2},$$

as this characterizes the fact that the Carleson Operator satisfies the desired bound as $\|\mathcal{C}f\|_{2,\infty} \leq C\|f\|_2$. From the classical theory of Lorentz and weak L^p spaces (see, for instance, [3, Chapter 1]), we know that the Fatou Inequality holds with these norms: that is, if $f_k \geq 0$, then $\| \liminf_{k \to \infty} f_k \|_{2,\infty} \leq \liminf_{k \to \infty} \|f_k\|_{2,\infty}$.

Now we use it with

$$\sup_\xi |\Pi_\xi f| \leq \liminf_{K,L \to \infty} \frac{1}{4KL} \int_{-L}^{L} \int_{-N}^{N} \int_0^1 \sup_{\xi \in \mathbb{R}} |M_{-\eta} T_{-y} D^2_{2^{-\lambda}} Q_{\frac{\xi+\eta}{2^\lambda}} D^2_{2^\lambda} T_y M_\eta f| \mathrm{d}y \, \mathrm{d}\eta \, \mathrm{d}\lambda,$$

along with the fact that the basic operators involved are all isometries with respect to the quasinorm $\|\cdot\|_{2,\infty}$ and the Minkowski's inequality for weak L^p spaces. This implies that, if we have an estimate of the kind

$$\|\sup_{\xi}|Q_\xi f|\|_{2,\infty}\le C\|f\|_2,$$

we can conclude.

STEP 2. Select a measurable function $N:\mathbb{R}\to\mathbb{R}$ such that, for all $x\in\mathbb{R}$,

$$\sup_{\xi}|Q_\xi f|\le 2|Q_{N(x)}f(x)|.$$

This is 'linearizing' the supremum, and that's what dubs our subsection. Again, it suffices to prove that $\|Q_N f\|_{2,\infty}\le C\|f\|_2$ to prove our result.

STEP 3. Instead of summing through all tiles, summing through a finite set will suffice to our purposes: actually, we could have taken the function N to take values only on (some finite subset of) the rational numbers. That would imply, in particular, that the series defining the function $Q_N f$ converges almost everywhere, and we may conclude that this sum is a limitting procedure of sums over finite sets tiles. Therefore, if, for a finite set of tiles $\mathbf{P}$, we have that $\|Q_N^{\mathbf{P}}f\|_{2,\infty}\le C\|f\|_2$ for C depending neither on N nor on $\mathbf{P}$, then we may take limits and again by Fatou's Lemma for the quasinorm we can conclude.

STEP 4. Now we employ another tool from the theory of rearrangement functions and Lorentz spaces: it is relatively easy to see that a function $g\in L^{2,\infty}$ if and only if it satisfies that, for all measurable set $E\subset\mathbb{R}$ with finite measure,

$$\left|\int_E g(x)\mathrm{d}x\right|\le C(g)|E|^{1/2}.$$

Apply it to the function $Q_{N(x)}f(x)$. We get almost what we want: for the Carleson operator to be of weak-type (2,2), it is enough that

$$\left|\sum_{s\in\mathbf{P}}\langle f,\phi_s\rangle\langle\phi_s,\chi_{E\cap N^{-1}(\omega_{s+})}\rangle\right|\le C\|f\|_2|E|^{1/2},$$

and now we perform a triangle inequality to get to the same expression as above, but with absolute values *inside* the sum.

STEP 5. The last one in our process. Now things reduce to only a change of variables: in fact, to reduce to $\|f\|_2=1$, we don't even have to do that, just divide by the norm. If, nevertheless, $2^k<|E|\le 2^{k+1}$, then let $E'=\frac{1}{2^k}E$ (set of points x such that $2^kx\in E$). This set has the required properties. But:

$$\begin{aligned}\int_{E\cap N^{-1}(\omega_{s+})}\phi_s(x)\mathrm{d}x &\overset{x=2^ky}{=} 2^k\int_{E'\cap\tilde{N}^{-1}(\omega_{s+})}\phi_s(2^ky)\mathrm{d}y\\ &= 2^{k/2}\int_{E'\cap(2^k\tilde{N})^{-1}(\omega_{s'})}\phi_{s'}(y)\mathrm{d}y,\end{aligned}$$

where $s' = (2^{-k}I_s) \times (2^k\omega_s)$ and $\tilde{N}(t) = N(2^k t)$. As $\phi_{s'}(x) = 2^{k/2}\phi_s(2^k x) = D^2_{2^k}\phi_s(x)$, we may, changing possibly f to $D^2_{2^{-k}}f$, E to E', $\mathbf{P}$ to $\mathbf{P}' = \{s'; s \in \mathbf{P}\}$ and N to $2^k\tilde{N}$, get to the desired conclusion that it is enough to prove for that specific class of objects. □

From now on, we let

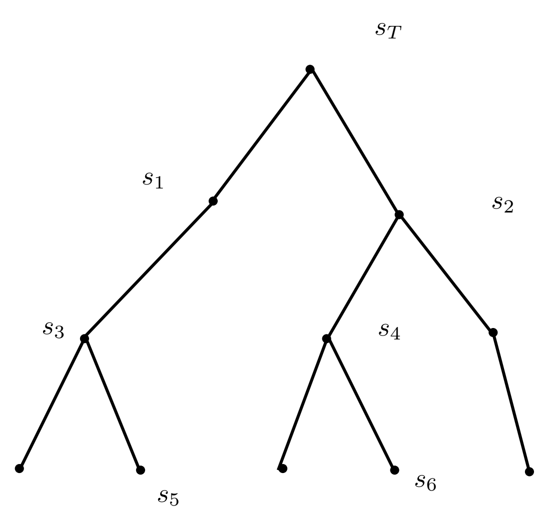
$$E_s = E \cap N^{-1}(\omega_s) \text{ and } E_{s+} = E \cap N^{-1}(\omega_{s+}).$$

3.4. **The Three Central Lemmas.** It turns out that the Key Estimate (3) has a lot to do with a way to order tiles. That is because the basic estimate can be even sharpened for a specific class of sets of tiles – which will be called trees –, and our task will be to decompose sets of tiles as unions of trees with nice properties.

Before going on with the proof, we have to start with the definitions:

Definition 1. *(a) We say that two tiles s, s' are* comparable *if $s \cap s' \neq \emptyset$. We write $s < s'$ if $I_s \subset I_{s'}$ and $\omega_{s'} \subset \omega_s$.*
(b) A set of tiles T is a tree *if there is a tile $s_T = I_T \times \omega_T$, which will be called the top of the tree, such that $s < s_T$ for all $s \in T$.*
(c) A tree is called a positive tree *(resp.,* negative tree*) if it satisfies that $\omega_{s+} \supset \omega_{s_T}$ (resp., $\omega_{s-} \supset \omega_{s_T}$) for every $s \in T$.*

The tree structure is quite simple to understand, and the basic drawing to illustrate this is the following:



Here, s_T is the top of our represented tree T, and $s_T > s_1 > s_3 > s_5$ and $s_T > s_2 > s_4 > s_6$.

These are the sets we are going to use to reduce the proof of the Key Estimate to another estimate, this time concerning trees. Before stating our Lemmas, we still have to define two very important notions:

Definition 2. *(i) We define the* mass *of a tile as the number*

$$M(\{s\}) = \sup_{s<p, p\in\mathcal{T}} \int_{E_p} w_p(x)\mathrm{d}x.$$

(ii) The mass of a set of tiles $\boldsymbol{P}$ *is just*

$$M(\boldsymbol{P}) = \sup_{s\in\boldsymbol{P}} M(\{s\}).$$

Definition 3. *(i) We define the* basic energy *of a Tree T as*

$$\Delta(T) = \left(\frac{1}{|I_T|}\sum_{s\in T} |\langle f, \phi_s\rangle|^2\right)^{\frac{1}{2}}.$$

(ii) The energy of a set of tiles $\boldsymbol{P}$ *is just*

$$E(\boldsymbol{P}) = \sup\{\Delta(\boldsymbol{T}); \boldsymbol{T}\subset\boldsymbol{P} \text{ positive tree }\}.$$

Let's state a few consequences of the definition. First, as $\|w_s\|_1 \leq 1$ for all tiles s, then so does $M(\mathbf{P}) \leq 1$, for all sets of tiles $\mathbf{P}$. Second, if $\mathbf{P}$ is *finite*, then there is at most a finite quantity of positive trees contained in it. As for finite trees the energy is always finite (because, for instante, of the Cauchy-Schwarz inequality), we see that the Energy of a set of tiles is always finite, but may depend on the finite set of tiles. However, this will not impose us a big restriction while proving the Key Estimate. Finally, we see directly from the definitions that Energy and Mass are both *monotonic* notions: if $\mathbf{P}\subset\mathbf{P}'$, then $E(\mathbf{P}) \leq E(\mathbf{P}')$ and $M(\mathbf{P}) \leq M(\mathbf{P}')$.

We now state the first two Lemmas we will use to prove the Theorem:

Lemma 3 (Mass Lemma)**.** *There is an absolute constant $C > 0$ such that, for every finite set of tiles $\boldsymbol{P}$, we may decompose it as $\boldsymbol{P} = \boldsymbol{P}_{light}\cup\boldsymbol{P}_{heavy}$, where*

$$M(\boldsymbol{P}_{light}) \leq \frac{1}{2}M(\boldsymbol{P}),$$

and, besides $M(\boldsymbol{P}_{heavy}) > \frac{1}{2}M(\boldsymbol{P})$, we can write $\boldsymbol{P}_{heavy} = \bigcup_{T\in\boldsymbol{T}} T$, where T are all trees, and

$$\sum_{T\in\boldsymbol{T}} |I_T| \leq \frac{C}{M(\boldsymbol{P})}.$$

What this Lemma is basically telling us is that every finite set of tiles has a subset that condenses most of its mass, and this set is well-behaved, in the sense that, if the mass of $\mathbf{P}$ is too big, then the elements of the set that concentrates mass $\mathbf{P}_{heavy}$ must be contained in 'narrow' trees. This may seem way too specific to care about, but we will see in a moment that it is what, in fact, will make it possible to estimate well the Key Expression in (3). In the same spirit, we have the following

Lemma 4 (Energy Lemma)**.** *There is an absolute constant $C > 0$ such that, for every finite set of tiles $\boldsymbol{P}$, we may decompose it as $\boldsymbol{P} = \boldsymbol{P}_{low}\cup\boldsymbol{P}_{high}$, where*

$$E(\boldsymbol{P}_{low}) \leq \frac{1}{2}E(\boldsymbol{P}),$$

and, besides $E(\boldsymbol{P}_{high}) > \frac{1}{2}E(\boldsymbol{P})$*, we can write* $\boldsymbol{P}_{high} = \bigcup_{T\in\boldsymbol{T}} T$*, where* T *are all trees, and*

$$\sum_{T\in\boldsymbol{T}} |I_T| \le \frac{C}{E(\boldsymbol{P})^2}.$$

We can already perform a decomposition of an arbitrary finite set of tiles $\mathbf{P}$ with these two Lemmas. We wish to decompose $\mathbf{P} = \cup_{n=-\infty}^{n_0}\mathbf{P}_n$, where $n_0 \in \mathbb{Z}$, and $\mathbf{P}_n$ has certain properties that make it possible for us to prove the Key Estimate. We then run the following algorithm:

(i) Pick the least $n \in \mathbb{Z}$ such that both $M(\mathbf{P}) \le 2^{2n}$ and $E(\mathbf{P}) \le 2^n$.
(ii) Select the inequality between the two above that is the 'borderline' one: that is, if $2^{n-1} < E(\mathbf{P}) \le 2^n$, select the second estimate, and if $2^{2(n-1)} < M(\mathbf{P}) \le 2^{2n}$, select the first one. Clearly, from the minimality of n, one of the two above must fulfill this requirement. For illustration purposes, we suppose it is the Energy estimate that is the borderline one.
(iii) Use the Energy (resp., Mass when we select the first estimate) Lemma, to decompose $\mathbf{P} = \mathbf{P}_{low} \cup \mathbf{P}_{high}$.
(iv) Add $\mathbf{P}_{high}$ to $\mathbf{P}_n$, and set $\mathbf{P}_{low} := \mathbf{P}$ on step (i).
(v) Repeat this process until we run out of tiles.

As our sets are all finite, our process does end. Moreover, we note that, from the way we have selected the sets that form $\mathbf{P}_n$, these sets can be written as $\mathbf{P}_n = \cup_{T\in\mathbf{T}_n} T$, where $\mathbf{T}_n$ is a set of trees, and they satisfy

$$\sum_{T\in\mathbf{T}_n} |I_T| \le C2^{-2n}.$$

This can be seen from the fact that all our inequalities selected are borderline ones, and that, from the selection and the Energy and Mass Lemmas, the sets $\mathbf{P'}$ selected in the process forming $\mathbf{P}_n$ satisfy either $E(\mathbf{P'}) \ge \frac{1}{4}2^n$ or $M(\mathbf{P'}) \ge \frac{1}{8}2^{2n}$. This, together with the tree structure of those sets, enables us to conclude the desired estimate.

To summarize, we have just proved the following

Proposition 5. *Every finite set of tiles* $\boldsymbol{P}$ *can be written as* $\bigcup_{n=-\infty}^{n_0} \boldsymbol{P}_n$*, where the sets* $\boldsymbol{P}_n$ *satisfy*

(a) $E(\boldsymbol{P}_n) \le 2^n$, $M(\boldsymbol{P}_n) \le 2^{2n}$.
(b) *Either* $E(\boldsymbol{P}_n) > \frac{1}{4}2^n$ *or* $M(\boldsymbol{P}_n) > \frac{1}{8}2^{2n}$.
(c) *The set* $\boldsymbol{P}_n = \bigcup_{T\in\boldsymbol{T}_n} T$ *of trees* T *such that*

$$\sum_{T\in\boldsymbol{T}_n} |I_T| \le C2^{-2n}.$$

As we promised, we now state – to only prove it later – the *Fundamental Estimate* that allows us to reduce the subject to trees.

Lemma 5 (Fundamental Estimate)**.** *There is an absolute constant* $C > 0$ *such that, for every tree* T*, the following inequality holds:*

$$\sum_{s\in \boldsymbol{T}} |\langle f,\phi_s\rangle\langle \phi_s,\chi_{E\cap N^{-1}(\omega_{s+})}\rangle| \leq CE(T)M(T)|I_T|.$$

Its proof is going to be exposed later, in the last section of this paper, because of its technical character. Nevertheless, this already permits us to prove the Key Estimate. First, decompose the set $\mathbf{P}$ according to the Proposition 5. Then we start to estimate: if we write

$$\begin{aligned}
\sum_{s\in \mathbf{P}} |\langle f,\phi_s\rangle\langle \phi_s,\chi_{E\cap N^{-1}(\omega_{s+})}\rangle| &\leq \sum_{n=-\infty}^{n_0}\left(\sum_{s\in \mathbf{P}_n} |\langle f,\phi_s\rangle\langle \phi_s,\chi_{E\cap N^{-1}(\omega_{s+})}\rangle|\right)\\
&\leq C\sum_{n=-\infty}^{n_0}\left(\sum_{T\in\mathbf{T}_n} E(T)M(T)|I_T|\right)\\
&\leq C\sum_{n=-\infty}^{n_0} 2^n\min(1,2^{2n})\left(\sum_{T\in\mathbf{T}_n}|I_T|\right)\\
&\leq C'\sum_{n=-\infty}^{n_0} 2^n\min(1,2^{2n})2^{-2n}\\
&\leq C'\sum_{n\in\mathbb{Z}}\min(2^n,2^{-n}) \leq 3C',
\end{aligned}$$

where (i) in the second inequality, we have used the decomposition of $\mathbf{P}_n$ and the Fundamental Estimate; (ii) in the third inequality, we have used the monotonicity of Energy and Mass and (iii) in the fourth inequality, we have used Proposition 5. So, for the rest of the paper, we are going to prove the Lemmas that we have used in the proof of the Key Estimate.

4. Proof of the Mass Lemma

We are going to decompose our finite set of tiles $\mathbf{P}$ into two sets. First, let $\mathbf{P}_{heavy}$ be *exactly* the set of tiles $\{s; M(\{s\}) > \frac{1}{2}M(\mathbf{P})\}$, and define the set $\mathbf{P}_{light}$ to be the set of all tiles in $\mathbf{P}$ that do not belong to $\mathbf{P}_{heavy}$. From this definition, we already see that the first part of the statement of the Mass Lemma already holds.Thus, it suffices to prove the second one.

For this purpose, let, for every tile $s\in\mathbf{P}$, $t(s)$ be some tile with $s < t(s)$ and

$$\int_{E_{t(s)}} w_{t(s)}(x)\mathrm{d}x > \frac{1}{2}M(\mathbf{P}).$$

Then form the collection $t(\mathbf{P}) = \{t(s); s\in\mathbf{P}\}$, and define $\mathbf{P}'$ to be the set of all tiles in $t(\mathbf{P})$ that are maximal with respect to the partial order of tiles.

Claim 1. *It suffices to prove that*

$$\sum_{s\in \boldsymbol{P}'} |I_s| \leq \frac{C}{M(\boldsymbol{P})},$$

for some absolute constant $C>0$.

Proof. If $\mathbf{P}_{heavy}$ is the union of a set of trees $\mathbf{T}$, then, if s_T and $s_{T'}$ are two top tiles subordinated to the same tile $s \in \mathbf{P}'$, we have that $I_T \cap I_{T'} = \emptyset$, because, as both are subordinated to s, $\omega_{s_T} \cap \omega_{s_{T'}} \supset \omega_s$. As both are tops, they cannot be comparable, and thus we have the desired disjointness. From this, we've that, collecting the tops of trees according to their superior tiles,

$$\sum_{T \in \mathbf{T}} |I_T| \leq \sum_{s \in \mathbf{P}'} |I_s|,$$

and then the claim follows easily. □

We are going to focus on the inequality stated on Claim 1. In order to prove it, we are going to divide the set $\mathbf{P}'$ into disjoint sets, each of which has nice properties:

Claim 2. *There exists $c > 0$ such that, for every tile $s \in \boldsymbol{P}'$, there exists $k \in \mathbb{N}$ such that*

$$|E_s \cap 2^k I_s| \geq c 2^{2k} M(\boldsymbol{P}) |I_s|.$$

Proof. We write

$$\sum_{k=1}^{\infty} \int_{E_s \cap 2^k I_s \setminus 2^{k-1} I_s} w_s(x) \mathrm{d}x = \int_{E_s} w_s(x) \mathrm{d}x > \frac{1}{2} M(\mathbf{P}).$$

From this, we may conclude that

$$\sum_{k=1}^{\infty} \frac{|E_s \cap 2^k I_s|}{|I_s|} (1 + 2^{k-1})^{-R} > \frac{1}{2} M(\mathbf{P}),$$

and, thus, there exists $k \in \mathbb{N}$ such that

$$\frac{|E_s \cap 2^k I_s|}{|I_s|} (1 + 2^{k-1})^{-R} > \frac{2^{-k}}{2} M(\mathbf{P}),$$

which, in turn, implies the desired inequality, with $c = 2^{-R-1}$ □

Inspired by the last claim, we decompose the set $\mathbf{P}'$ into the sets $\mathbf{P}'_k$, each of which consists of the tiles in $\mathbf{P}'$ such that the inequality in Claim 2 is fulfilled with k being the least possible. This allows us to write $\mathbf{P}' = \cup_{k \in \mathbb{N}} \mathbf{P}'_k$. Obviously, in order to prove the inequality (1), it is enough to prove that

$$\sum_{s \in \mathbf{P}'_k} |I_s| \leq C 2^{-k} M(\mathbf{P})^{-1}.$$

To prove this last inequality, and finally complete the proof of the Mass Lemma, we perform a Vitali-like process of selection of tiles: Build, in $\mathbf{P}'_k$, the enlarged rectangles $\tilde{s} = 2^k I_s \times \omega_s$. Our proccess will, then, select successively the tile in $\mathbf{P}'_k$ with the largest $|I_s|$ and delete all tiles such that their enlarged rectangles intersect the enlarged rectangle of the selected tile.

This produces a refined collection of tiles $\mathbf{P}''_k$ that satisfies the condition that, for all $s \in \mathbf{P}'_k$, it is associated to a *unique* tile $s' \in \mathbf{P}''_k$ with the property that $|I_s| \leq |I_{s'}|$, and the enlarged rectangles $\tilde{s} \cap \tilde{s'} \neq \emptyset$. For a fixed element $s \in \mathbf{P}''_k$, as the tiles in $\mathbf{P}'$ are pairwise disjoint, but the enlarged rectangles of all tiles associated to s are not, we conclude that, if p, q are tiles, both of which are associated to s, then $I_p \cap I_q = \emptyset$, and $I_p, I_q \subset 2^{k+2} I_s$. Therefore,

$$\sum_{s \in \mathbf{P}'_k} |I_s| \le \sum_{s' \in \mathbf{P}''_k} 2^{k+2} |I_{s'}| \le C 2^{-k} M(\mathbf{P})^{-1} \sum_{s' \in \mathbf{P}''_k} |E_{s'} \cap 2^k I_{s'}| \le C' 2^{-k} M(\mathbf{P})^{-1},$$

where the second inequality is justified by the definition of the set $\mathbf{P}'_k$, and the third one by the disjointness properties of the enlarged rectangles in the refined collection. This completes the proof of the Mass Lemma.

5. Proof of the Energy Lemma

The basic idea to prove this lemma is essentially the same one in the Mass Lemma: we are going to pick undesirable trees and remove them from our collection. More specifically, we adopt the following procedure: we select the positive tree T with the property that

(i) $\Delta(T) > \frac{1}{2} E(\mathbf{P})$;

(ii) $c(\omega_T)$ is minimal among the positive trees with the desired property.

After picking the positive tree T, we select the biggest tree T^1 inside $\mathbf{P}$ with the same top as T. We then add T to $\mathbf{T}_+$ and T^1 to $\mathbf{T}$, and repeat the strategy above until we run out of choices of positive trees. In this case, either there will be no tile left, in which case we set $\mathbf{P}_{low} = \emptyset$, or there are going to be some tiles left, and we then take them as our $\mathbf{P}_{low}$. We see straightforwardly that the set $\mathbf{P}_{low}$ satisfies the first assertion of the Energy Lemma. As it is expected, the set $\mathbf{P}_{high}$ is defined as the union of the trees in $\mathbf{T}$.

Before proving the second assertion, we need the following

Claim 3 (Strong Disjointness Property)**.** *(i) Suppose that $s \in T, s' \in T'$, where $T \neq T'$ are positive trees in $\mathbf{T}_+$. If $\omega_s \subset \omega_{s'-}$, then $I_T \cap I_{s'} = \emptyset$.*

(ii) Suppose that $p \in T_1, q \in T_2$ are two tiles, with $T_1 \neq T_2$ being positive trees in $\mathbf{T}_+$, such that $\omega_s \subset \omega_{p-} \cap \omega_{q-}$.Then $I_p \cap I_q = \emptyset$.

Proof. (i) $c(\omega_T) \in \omega_T \subset \omega_s \subset \omega_{s'-} \Rightarrow c(\omega_T) < c(\omega_{T'})$, and, therefore, T has been selected before T'. If $I_T \cap I_{s'} \neq \emptyset$, then $s' \in T^1$, which is impossible, as this implies that s' would have to be selected *before* T'.

(ii) Without loss of generality, $\omega_s \subset \omega_{p-} \subset \omega_{q-} \Rightarrow \omega_p \subset \omega_{q-}$. This, along with part (i), implies that $I_p \cap I_q \subset I_{T_1} \cap I_q = \emptyset$, and this completes the proof of the claim. □

Now, we are ready to prove the Lemma. We have to prove that

$$E(\mathbf{P})^2 \sum_{T \in \mathbf{T}_2} |I_T| \le C, \tag{4}$$

for some absolute constant $C > 0$. Let, now, $\tilde{\mathbf{P}}$ be the set of tiles in $\cup_{T \in \mathbf{T}_2} T$. A simple calculation shows that the left hand side of (4) is bounded by a constant times

$$\sum_{s \in \tilde{P}} |\langle f, \phi_s \rangle|^2,$$

which, in turn, is bounded by

$$\left\| \sum_{s \in \tilde{P}} \langle f, \phi_s \rangle \phi_s \right\|_2,$$

because $\|f\|_2 = 1$. To prove the desired bound, it suffices to prove that

$$\left\| \sum_{s \in \tilde{P}} \langle f, \phi_s \rangle \phi_s \right\|_2^2 \le C E(\mathbf{P})^2 \sum_{T \in \mathbf{T}_2} |I_T| \tag{5}$$

Because of the fact that $\langle \phi_s, \phi_{s'} \rangle \neq 0 \Rightarrow \omega_{s-} \cap \omega_{s'-} \neq \emptyset$, using the symmetry involved, we may bound the required left hand side of 5 by

$$\sum_{s,s' \in \tilde{\mathbf{P}}; \omega_s = \omega_{s'}} |\langle f, \phi_s \rangle \langle \phi_s, \phi_{s'} \rangle \langle f, \phi_{s'} \rangle| + 2 \sum_{s,s' \in \tilde{\mathbf{P}}; \omega_s \subset \omega_{s'-}} |\langle f, \phi_s \rangle \langle \phi_s, \phi_{s'} \rangle \langle f, \phi_{s'} \rangle|. \tag{6}$$

In order to continue, we need an estimate concerning the inner product of functions adapted to tiles:

Lemma 6. *For s, s' tiles such that $|I_{s'}| \le |I_s|$, we have that*

$$|\langle \phi_s, \phi_{s'} \rangle| \le C |I_s|^{1/2} |I_{s'}|^{-1/2} \|w_s \chi_{s'}\|_1.$$

Proof. First, we are going to prove that, if $|I_{s'}| \le |I_s|$, then, in the same condition as above,

$$|\langle \phi_s, \phi_{s'} \rangle| \le C |I_{s'}|^{1/2} |I_s|^{-1/2} \left(1 + \frac{|c(I_{s'}) - c(I_s)|}{|I_s|}\right)^{-R}$$

We distinguish two cases: First, we are going to deal with the case when $|c(I_s) - c(I_{s'})| \le |I_s|$. In this case, we simply use Hölder's inequality:

$$\begin{aligned} |\langle \phi_s, \phi_{s'} \rangle| \le \|\phi_{s'}\|_1 \|\phi_s\|_\infty &\le C |I_{s'}|^{1/2} |I_s|^{-1/2} \\ &\le \tilde{C} |I_{s'}|^{1/2} |I_s|^{-1/2} \left(1 + \frac{|c(I_{s'}) - c(I_s)|}{|I_s|}\right)^{-R}. \end{aligned}$$

For the second one, supose, without loss of generality, that $c(I_{s'}) < c(I_s)$, and let c be the midpoint of them. We split

$$
\begin{aligned}
|\langle \phi_s, \phi_{s'} \rangle| &\leq \left| \int_{-\infty}^{c} \phi_s(x)\overline{\phi_{s'}(x)}\mathrm{d}x \right| + \left| \int_{c}^{+\infty} \phi_s(x)\overline{\phi_{s'}(x)}\mathrm{d}x \right| \\
&\leq \|\phi_{s'}\|_1 \|\phi_s\|_{L^\infty(-\infty,c)} + \|\phi_s\|_1 \|\phi_{s'}\|_{L^\infty(c,+\infty)} \\
&\leq C|I_{s'}|^{1/2}|I_s|^{-1/2} \left(1 + \frac{|c - c(I_s)|}{|I_s|}\right)^{-R} + \\
&\quad C|I_s|^{1/2}|I_{s'}|^{-1/2} \left(1 + \frac{|c - c(I_{s'})|}{|I_{s'}|}\right)^{-R} \\
&\leq C|I_{s'}|^{1/2}|I_s|^{-1/2} \left(1 + \frac{|c(I_{s'}) - c(I_s)|}{|I_s|}\right)^{-R},
\end{aligned}
$$

Where (i) the second inequality is just Hölder's inequality; (ii) in the third one, we use the estimate $|\phi_s(x)| \leq C|I_s|^{1/2}w_s(x)$; (iii) in the fourth, we notice that c is the midpoint between $c(I_s)$ and $c(I_{s'})$, and simply estimate the second summand by the first (as R is large).

Now, to prove the desired original inequality, we notice that, if $x \in I_{s'}$, then, from triangle's inequality,

$$
\left| \frac{|x - c(I_s)|}{|I_s|} - \frac{|c(I_s) - c(I_{s'})|}{|I_s|} \right| \leq \frac{1}{2}.
$$

We get almost directly from that that there are $c_R, C_R > 0$ such that, for all $x \in I_{s'}$,

$$
C_R \left(1 + \frac{|c(I_s) - c(I_{s'})|}{|I_s|}\right)^{-R} \geq \left(1 + \frac{|x - c(I_s)|}{|I_s|}\right)^{-R} \geq c_R \left(1 + \frac{|c(I_s) - c(I_{s'})|}{|I_s|}\right)^{-R}
$$

This, along with the previous estimate, allows us to conclude the proof of the claim □

This estimate already allows us to successfully estimate the first summand of (6): Indeed, we first use a Cauchy-Schwarz inequality in the sum, estimate the terms concerning $|\langle \phi_s, \phi_{s'} \rangle|$, use the disjointness of the intervals I_p such that $w_p = w_{p'}$ and the fact that $\|w_s\|_1 = 1$ to conclude:

$$
\begin{aligned}
\sum_{s,s' \in \tilde{\mathbf{P}}; \omega_s = \omega_{s'}} |\langle f, \phi_s \rangle \langle \phi_s, \phi_{s'} \rangle \langle f, \phi_{s'} \rangle| &\leq C \sum_{s,s' \in \tilde{\mathbf{P}}; \omega_s = \omega_{s'}} |\langle f, \phi_s \rangle|^2 |\langle \phi_s, \phi_{s'} \rangle| \\
&\leq C \sum_{s \in \tilde{\mathbf{P}}} |\langle f, \phi_s \rangle|^2 \left(\sum_{s' \in \tilde{\mathbf{P}}; \omega_s = \omega_{s'}} \|w_s \chi_{I_{s'}}\|_1 \right) \\
&\leq C \sum_{s \in \tilde{\mathbf{P}}} |\langle f, \phi_s \rangle|^2 \\
&\leq C E(\mathbf{P})^2 \sum_{T \in \mathbf{T}_+} |I_T|,
\end{aligned}
$$

as the last estimate follows from the definition of $\tilde{\mathbf{P}}$. The second part of the estimate is subtler. We begin by estimating it, with the use of a Cauchy-Schwarz inequality, by

$$\sum_{T\in\mathbf{T}_+}\left(\sum_{s\in T}|\langle f,\phi_s\rangle|^2\right)^{1/2}H(T)^{1/2}\leq CE(\mathbf{P})\sum_{T\in\mathbf{T}_+}|I_T|^{1/2}H(T)^{1/2},$$

where we define

$$H(T):=\sum_{s\in T}\left(\sum_{s'\in\tilde{\mathbf{P}};\omega_s\subset\omega_{s'\_}}|\langle\phi_s,\phi_{s'}\rangle||\langle f,\phi_{s'}\rangle|\right)^2.$$

This reduces the matter to showing that $H(T)\leq CE(\mathbf{P})^2|I_T|$. In order to achieve this, we first observe that $\{s\}$ is, itself, a positive tree, and therefore it satisfies $|\langle f,\phi_s\rangle|\leq E(\mathbf{P})|I_s|^{1/2}$. Thus, using also the almost orthogonality Lemma above, we get that

$$H(T)\leq CE(\mathbf{P})^2\sum_{s\in T}|I_s|\left(\sum_{s'\in\tilde{\mathbf{P}};\omega_s\subset\omega_{s'\_}}\|w_s\chi_{I_{s'}}\|_1\right)^2.$$

Now the Strong Disjointness Property becomes useful. Notice that, from this property, all the intervals $I_{s'}$ in the sum inside the parenthesis are mutually disjoint. Moreover, they are all disjoint from I_T. Therefore, we may bound this sum by

$$\sum_{s'\in\tilde{\mathbf{P}};\omega_s\subset\omega_{s'\_}}\|w_s\chi_{I_{s'}}\|_1\leq C\|w_s\chi_{I_T^c}\|_1.$$

As we also have that $\|w_s\|_1=1\Rightarrow\|w_s\chi_{I_T^c}\|_1^2\leq\|w_s\chi_{I_T^c}\|$, and this permits us to estimate

$$H(T)\leq CE(\mathbf{P})^2\sum_{s\in T}|I_s|\|w_s\chi_{I_T^c}\|_1.$$

To bound $\sum_{s\in T}|I_T|\|w_s\chi_{I_T^c}\|_1$, we notice that, for each $k\in\mathbb{N}$, we may divide the interval I_T into 2^k intervals of length $|I_T|2^{-k}$. For each of these, there is at most one interval $s\in T$ such that the spatial component of s, I_s is a specific tile in the $k-$scale of I_T, because, if s,s' are both tiles with $I_s=I_{s'}$, then, as $\omega_s\cap\omega_{s'}\neq\emptyset$, $s=s'$. Then:

$$\begin{aligned}\sum_{s\in T}|I_s|\|w_s\chi_{I_T^c}\|_1&\leq\sum_{k=1}^{\infty}\frac{2^k}{|I_T|}\sum_{s\in T;|I_s|=2^{-k}|I_T|}|I_s|\int_{I_T^c}\frac{\mathrm{d}x}{(1+\frac{|x-c(I_s)|}{|I_s|})^R}\\&\leq C_1\sum_{k=1}^{\infty}\frac{2^k}{|I_T|}\sum_{s\in T;|I_s|=2^{-k}|I_T|}\int_{I_s}\int_{I_T^c}\frac{\mathrm{d}x\mathrm{d}y}{(1+\frac{|x-y|}{|I_s|})^R}\\&\leq C_2\sum_{k=1}^{\infty}\frac{2^k}{|I_T|}\int_{I_T}\int_{I_T^c}\frac{\mathrm{d}x\mathrm{d}y}{(1+\frac{|x-y|}{|I_s|})^R}\\&\leq C_3|I_T|,\end{aligned}$$

where, in the first two inequalities, we have used the aforementioned disjointness of space-scales, and the following result:

Lemma 7. *For every interval $J \subset \mathbb{R}$ and every real number $b > 0$, we have that*

$$\int_J \int_{J^c} \frac{\mathrm{d}x\mathrm{d}y}{(1 + \frac{|x-y|}{b|J|})^R} \leq C(b|J|)^2$$

Proof. We may, after translating, suppose that $J = [0, |J|]$, and, after changing variables, that $|J| = 1/b$. In this case, we reduce matters to proving that

$$\int_{[0,1/b]} \int_{[0,1/b]^c} \frac{\mathrm{d}x\mathrm{d}y}{(1 + |x-y|)^R} \leq C.$$

But this is explicitly computable; let's evaluate only the part $\int_{1/b}^{\infty}$:

$$\int_{1/b}^{\infty} \frac{\mathrm{d}y}{(1+y-x)^R} = \int_{1/b-x}^{\infty} \frac{\mathrm{d}t}{(1+t)^R} = \frac{(1+1/b-x)^{1-R}}{R-1},$$

and, thus, our desired integral is

$$\int_0^{1/b} \frac{\mathrm{d}x}{(R-1)(1+t)^{R-1}} = \frac{1-(1+1/b)^{2-R}}{(R-1)(R-2)} \leq 1,$$

for a suitably large R. □

From this last inequality, we prove the desired bound on $H(T)$, which completes the proof of the Energy Lemma.

6. Proof of the Fundamental Estimate Lemma

Let $\mathcal{J}'$ be the colection of dyadic intervals of scale $\frac{1}{4}l(T)$, where $l(T) = \inf\{|I_s|, s \in T\}$. For all $J' \in \mathcal{J}'$, $3J'$ does not contain any of the intervals I_s. Now let $\mathcal{J}$ be the colection of *maximal* dyadic intervals with this property. Clearly, $\mathcal{J}$ constitutes a partition of $\mathbb{R}$.

Now we write

$$\sum_{s\in T} |\langle f, \phi_s\rangle\langle\phi_s, \chi_{E_{s+}}\rangle| = \sum_{s\in T} \epsilon_s \langle f, \phi_s\rangle\langle\phi_s, \chi_{E_{s+}}\rangle,$$

where ϵ_s are complex scalars of absolute value 1. We then reduce the estimate on the sum above to an estimate on

$$\begin{aligned}\left\|\sum_{s\in T} \epsilon_s\langle f,\phi_s\rangle\phi_s\chi_{E_{s+}}\right\|_1 &\leq \sum_{J\in\mathcal{J}}\left(\sum_{s\in T;|I_s|\leq 2|J|} \|\langle f,\phi_s\rangle\phi_s\chi_{E_{s+}}\|_{L^1(J)}\right) \\ &\quad + \sum_{J\in\mathcal{J}}\left\|\sum_{s\in T;|I_s|>2|J|} \epsilon_s\langle f,\phi_s\rangle\phi_s\chi_{E_{s+}}\right\|_{L^1(J)} \\ &=: S_1 + S_2.\end{aligned}$$

We are going to sepparately estimate S_1 and S_2 next:

6.1. **Estimating S_1.** We begin by noting that, from the Energy definition and the fact that $\phi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\|\langle f,\phi_s\rangle\phi_s\chi_{E_{s+}}\|_{L^1(J)} &\le CE(T)|I_s|^{1/2}|I_s|^{1/2}\|w_s^2\chi_{E_{s+}}\|_{L^1(J)}\\ &\le CE(T)M(T)|I_s|\left(1+\frac{\operatorname{dist}(I_s,J)}{|I_s|}\right)^{-R}.\end{aligned}$$

Now we break the first sum defining S_1 into scales:

$$\begin{aligned}&\sum_{s\in T;|I_s|\le 2|J|}\|\langle f,\phi_s\rangle\phi_s\chi_{E_{s+}}\|_{L^1(J)}\le\\ &\le CE(T)M(T)\sum_{k\le\log_2 2|J|}\left(\sum_{s\in T;|I_s|=2^k}2^k\left(1+\frac{\operatorname{dist}(I_s,J)}{|I_s|}\right)^{-R}\right)\\ &\le CE(T)M(T)\sum_{k\le\log_2 2|J|}2^k\left(\sum_{s\in T;|I_s|=2^k}\left(1+\frac{\operatorname{dist}(I_T,J)}{|I_T|}\right)^{-R/2}\left(1+\frac{\operatorname{dist}(I_s,J)}{|I_s|}\right)^{-R/2}\right)\\ &\le C'E(T)M(T)\sum_{k\le\log_2 2|J|}2^k\left(1+\frac{\operatorname{dist}(I_T,J)}{|I_T|}\right)^{-R/2},\end{aligned}$$

where the second inequality follows from the fact that $\operatorname{dist}(I_T,J)/|I_T|\le\operatorname{dist}(I_s,J)/|I_s|$, and the third one from the fact that, as $3J$ does not contain any I_s, there are at most two intervals I_s such that neither of them are contained in $3J$ and $\operatorname{dist}(I_s,J)\in[m2^k,(m+1)2^k)$. This last estimate plainly implies that

$$\begin{aligned}S_1&\le C'E(T)M(T)\sum_{J\in\mathcal{J}}|J|\left(1+\frac{\operatorname{dist}(I_T,J)}{|I_T|}\right)^{-R/2}\\ &\le C'E(T)M(T)\int_{\mathbb{R}}\left(1+\frac{|x-c(I_T)|}{|I_T|}\right)^{-R/2}\mathrm{d}x\\ &\le C'E(T)M(T)|I_T|,\end{aligned}$$

where our second inequality followed from the fact that either $J\cap I_T=\emptyset$ or $J\subset I_T$, and these imply, in turn, that, after some boresome calculations,

$$\left(1+\frac{\operatorname{dist}(I_T,J)}{|I_T|}\right)^{-R/2}\le C_R\left(1+\frac{|x-c(I_T)|}{|I_T|}\right)^{-R/2},\ \forall x\in J,$$

and we used that $\mathcal{J}$ partitions $\mathbb{R}$. This finishes this part of the proof.

6.2. **Estimating S_2.** Notice that we might reduce the sum defining S_2 to one that contains only those intervals J such that $2|J|<|I_s|$ for some s. In this case, $|J|<|I_T|$ obviously, and then $J\subset 3I_T$ (as all dyadic intervals in $\mathcal{J}$ disjoint from $3I_T$ may be taken – as they are maximal – with lengths $\ge|I_T|$). Then, we define $T_+=\{s\in T;\omega_{s+}\supset\omega_{T+}\}$, and $T_-=T\setminus T_+$. Related to these two trees we define the functions

$$F_{J\pm} := \sum_{s\in T_\pm;|I_s|>2|J|} \epsilon_s\langle f,\phi_s\rangle\phi_s\chi_{E_{s+}}.$$

Clearly, estimating S_2 is the same, after a triangle inequality, as estimating these two kinds of functions. That is what we are going to do:

6.2.1. *Inequality for F_{J-}.* We notice first that, whenever two tiles s, s' belong to different scales, then either $\omega_s \subset \omega_{s'-}$ or the reverse inclusion holds. This already implies that $\omega_{s+}\cap\omega_{s'+} = \emptyset$ for those tiles, and consequently $E_{s+}\cap E_{s'+} = \emptyset$. Therefore,

$$\begin{aligned}
\|F_{J-}\|_{L^1(J)} &\le \|F_{J-}\|_{L^\infty(J)}|G(J)| \\
&\le \sup_{k>\log_2 2|J|}\sup_{x\in J}\left|\sum_{s\in T_-;|I_s|=2^k}\epsilon_s\langle f,\phi_s\rangle\phi_s(x)\chi_{E_{s+}}\right||G(J)| \\
&\le \sup_{k>\log_2 2|J|}\sup_{x\in J}\sum_{s\in T_-;|I_s|=2^k}E(T)\frac{1}{\left(1+\frac{|x-c(I_s)|}{2^k}\right)^R}|G(J)| \\
&\le CE(T)\sup_{k>\log_2 2|J|}\sup_{x\in J}2^{-k}\int_{\cup I_s}\frac{\mathrm{d}y}{\left(1+\frac{|x-y|}{2^k}\right)^R}|G(J)| \\
&\le C'E(T)|G(J)|,
\end{aligned}$$

where we define $G(J) = J\cap\bigcup_{s\in T;|I_s|>2|J|}E_{s+}$. The second inequality is justified by the mentioned disjointness of scales, the third one by the fact that $|\phi_s(x)| \le C|I_s|^{1/2}w_s(x)$ and that a single tile $\{s\}$ is itself a positive tree, the fourth one by estimating the terms $\frac{1}{\left(1+\frac{|x-c(I_s)|}{2^k}\right)^R}$ by the respective integrals over I_s and the fifth one by a direct integration after a change of variables. To finish this term, we have to prove the following

Lemma 8. *There is $C>0$ such that, for all $J\in\mathcal{J}$, we have*

$$|G(J)| \le CM(T)|J|.$$

Proof. Let J' be the dyadic interval with $|J'| = 2|J| \le |I_T|$. By maximality of J, we have that $3J' \supset I_{s_0}$ for some $s_0\in T$. There may be some possibilites for s_0 inside the set $3J'$, but what matters is that there is a tile s' with either $|I_{s'}| = |J'|$ or $|I_{s'}| = 2|J'|$ – depending on the properties of I_{s_0} – such that $s < s' < s_T$, and $I_{s'} \subset 3J'$. We then claim that $G(J)\subset J\cap E_{s'}$. Indeed, $\omega_{s'}\cap\omega_s \ne \emptyset$ for all $s\in T$ such that $|I_s| > 2|J|$. But then $|\omega_s| \le \frac{1}{4|J|} \le \frac{1}{|I_{s'}|} = |\omega_{s'}|$. This implies that $\omega_{s'}\supset\omega_s$, and, after remembering that $E_{s+} = E\cap N^{-1}(\omega_{s+}) \subset E\cap N^{-1}(\omega_{s'}) = E_{s'}$, we prove our claim. To finish, we just estimate

$$\begin{aligned}|G(J)| &\le \left|\int \chi_J \chi_{E_{s'}} \mathrm{d}x\right| \\ &\le C_N |I_{s'}| \left|\int \chi_J w_{s'}(x) \chi_{E_{s'}} \mathrm{d}x\right| \\ &\le C'|J| M(T),\end{aligned}$$

where we have used that, for $x \in J$,

$$|x - c(I_{s'})| \le 10|J| \le 5|I_{s'}| \Rightarrow 1 \le 6^R \left(1 + \frac{|x - c(I_{s'})|}{|I_{s'}|}\right)^{-R},$$

□

Now we are able to finish our inequality concerning F_{J-}:

$$\sum_{J \in \mathcal{J}; J \subset 3I_T} \|F_{J-}\|_{L^1(J)} \le CE(T)M(T) \sum_{J \in \mathcal{J}; J \subset 3I_T} |J| \le 3CE(T)M(T)|I_T|.$$

6.2.2. *Inequality for F_{J+}.* This is where all the difficulty relies.

Suppose that, for $x \in J \in \mathcal{J}$ we have that $F_{J+}(x) \ne 0$. Then, as the intervals $\{\omega_s, s \in T_+\}$ intersect, we may pick the largest interval $\omega_{x,1}$ of the form ω_s, where $s \in T_+$, besides $x \in E_{s+}$ and $|I_s| > 2|J|$. In the same way, we pick the smallest interval $\omega_{x,2}$ of the form ω_{s+} such that these conditions hold. Then it is straightforward to see that a term in the sum defining F_{J+} is nonzero if and only if $|\omega_{x,2}| < |\omega_s| \le |\omega_{x,1}|$. We write, then:

$$F_{J+}(x) = \sum_{s \in T_+; |\omega_{x,2}| < |\omega_s| \le |\omega_{x,1}|} \epsilon_s \langle f, \phi_s \rangle \phi_s(x).$$

Now let $\psi \in \mathcal{S}(\mathbb{R})$ be a function such that $\chi_{(-\frac{1}{2},\frac{1}{2})} \le \widehat{\psi} \le \chi_{(-\frac{1}{2}-\frac{1}{20},\frac{1}{2}+\frac{1}{20})}$.

Claim 4. *For every $s \in T_+$, we have that*

(a) *if s accounts for the sum defining F_{J+} above, then*

$$\phi_s = \phi_s * (M_{c(\omega_{x,1})} D^1_{|\omega_{x,1}|^{-1}} \psi - M_{c(\omega_{x,2})} D^1_{|\omega_{x,2}|^{-1}} \psi).$$

(b) *if s does* not *account for the sum above, then*

$$0 = \phi_s * (M_{c(\omega_{x,1})} D^1_{|\omega_{x,1}|^{-1}} \psi - M_{c(\omega_{x,2})} D^1_{|\omega_{x,2}|^{-1}} \psi).$$

Proof. (a) We take Fourier transforms on both sides, and notice that – as Fourier transforms take convolution products to regular products – it suffices to prove that

$$\widehat{\psi}\left(\frac{\xi - c(\omega_{x,1})}{|\omega_{x,1}|}\right) - \widehat{\psi}\left(\frac{\xi - c(\omega_{x,2})}{|\omega_{x,2}|}\right) \equiv 1$$

on the support of $\widehat{\phi_s} \subset \frac{1}{5}\omega_{s-}$. But then, for $\xi \in$ supp $\widehat{\phi_s}$, $|\xi - c(\omega_{x,1})| < \frac{1}{2}|\omega_{x,1}|$, because $\omega_s \subset \omega_{x,1}$ in this case. Thus, the first summand is 1. It remains to show that the second is zero. But

$$|\xi - c(\omega_{x,2})| \ge \mathrm{dist}(\frac{1}{5}\omega_{s-}, c(\omega_{x,2})) \ge \frac{1}{4}|\omega_s| + \mathrm{dist}(\frac{1}{5}\omega_{s-}, \omega_{s+}) = \frac{9}{20}|\omega_s| \ge |\frac{18}{20}|\omega_{x,2}|,$$

and this implies that the second summand must be zero. This already gives us the desired equality.

(b) We will do the case where $|\omega_s| > |\omega_{x,1}|$, the other one being similar. In this case, the second summand is still zero, but $\omega_{x,1} \subset \omega_{s+}$:

$$|\xi - c(\omega_{x,1})| \geq \text{dist}(\frac{1}{5}\omega_{s-}, c(\omega_{x,1})) \geq \frac{1}{4}|\omega_s| + \text{dist}(\frac{1}{5}\omega_{s-}, \omega_{s+}) = \frac{9}{20}|\omega_s| \geq \frac{18}{20}|\omega_{x,1}|,$$

and this shows also that the first summand must be zero. □

Write, then

$$F_{J+}(x) = \left[\sum_{s\in T_+} \epsilon_s \langle f, \phi_s\rangle \phi_s\right] * (M_{c(\omega_{x,1})} D^1_{|\omega_{x,1}|^{-1}}\psi - M_{c(\omega_{x,2})} D^1_{|\omega_{x,2}|^{-1}}\psi)(x).$$

To shorten notation, let

$$F(x) = \sum_{s\in T_+} \epsilon_s \langle f, \phi_s\rangle \phi_s.$$

Then we may estimate F_{J+} by

$$\begin{aligned} 2 \sup_{t\geq |\omega_{x,1}|^{-1}} \frac{1}{t}\int |F(y)| \left|\psi\left(\frac{x-y}{t}\right)\right| \mathrm{d}y &\leq C \sup_{t\geq |\omega_{x,1}|^{-1}} \frac{1}{t}\int |F(y)| w\left(\frac{x-y}{t}\right) \mathrm{d}y \\ &\leq C \sup_{t > 2|J|} \frac{1}{t}\int |F(y)| w\left(\frac{x-y}{t}\right) \mathrm{d}y \\ &\leq C \sup_{J\subset I} \langle |F|, w_I\rangle, \end{aligned}$$

because $|\omega_{x,1}|^{-1} = |I_{q_0}| > 2|J|$ and $B(x,\alpha) \supset J$ if $\alpha > 2|J|, x \in J$. Notice that this last expression is constant on J. We are now ready to estimate:

$$\begin{aligned} \sum_{J\in\mathcal{J}} \|F_{J+}\|_{L^1(J)} &\leq C \sum_{J\in\mathcal{J}; J\subset 3I_T} |G(J)| \sup_{J\subset I}\langle |F|, w_I\rangle \\ &\leq C' \sum_{J\in\mathcal{J}; J\subset 3I_T} M(T) \int_J \sup_{J\subset I}\langle |F|, w_I\rangle \mathrm{d}y \\ &\leq C' M(T) \int_{3I_T} \sup_{J\subset I}\langle |F|, w_I\rangle \mathrm{d}y \\ &\leq \tilde{C} M(T) \int_{3I_T} MF(y) \mathrm{d}y \\ &\leq \tilde{C}' M(T) \|MF\|_2 |I_T|^{1/2} \leq \tilde{C}'' M(T) \|F\|_2 |I_T|^{1/2}, \end{aligned} \tag{7}$$

where M denotes the Hardy-Littlewood Maximal operator. Here we have used a similar idea from the proof of Proposition 3. It suffices to use the following lemma to finish:

Lemma 9. *Let T_+ be a positive tree, and take $\{\lambda_s\}_{s\in T_+}$ to be a sequence of Complex Numbers. Then*

$$\left\| \sum_{s \in T_+} \lambda_s \phi_s \right\|_2 \leq C_1 \left(\sum_{s \in T_+} |\lambda_s|^2 \right)^{1/2}.$$

Proof. This lemma is close in spirit to some propositions we have already seen. We calculate

$$\begin{aligned}
\left\| \sum_{s \in T_+} \lambda_s \phi_s \right\|_2^2 &= \sum_{s,s' \in T_+} \lambda_s \overline{\lambda_{s'}} \langle \phi_s, \phi_{s'} \rangle \\
&= \sum_{s,s' \in T_+; \omega_s = \omega_{s'}} \lambda_s \overline{\lambda_{s'}} \langle \phi_s, \phi_{s'} \rangle \\
&\leq \sum_{s,s' \in T_+; \omega_s = \omega_{s'}} |\lambda_s|^2 |\langle \phi_s, \phi_{s'} \rangle| \\
&\leq \sum_{s \in T_+} \left(\sum_{s' \in T_+; \omega_s = \omega_{s'}} |\langle \phi_s, \phi_{s'} \rangle| \right) |\lambda_s|^2 \\
&\leq C \sum_{s \in T_+} |\lambda_s|^2,
\end{aligned}$$

where we have used the fact that the bottom parts are disjoint if tiles are from different scales – as the tree is positive –, a Cauchy-Schwarz inequality and Lemma 6. For further details, see the proof of Proposition 3. □

Now, we use it with the definition of F. We get immediately that

$$\|F\|_2 \leq C_1 \left(\sum_{s \in T_+} |\langle f, \phi_s \rangle|^2 \right)^{1/2} \leq C_1 E(T) |I_T|^{1/2}.$$

Put toghether with the previous estimate (7), we finish the proof of the Fundamental Estimate.

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