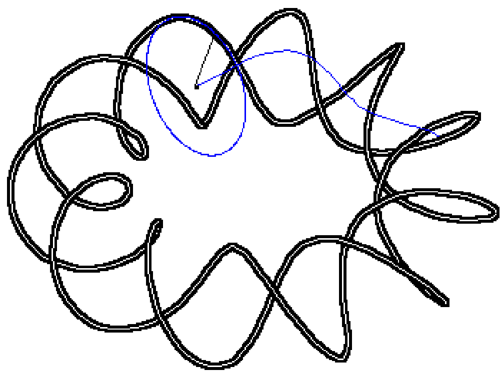
Space Curves of Constant Curvature *



2 - 11 Torus Knot of constant curvature.

See also:
About Spherical Curves

Definition via Differential Equations. Space Curves that 3DXM can exhibit are mostly given in terms of explicit formulas or explicit geometric constructions. The differential geometric treatment of curves starts from such examples and defines geometric properties, i.e., properties which do not change when the curve is transformed by an isometry (= distance preserving map, also called a rigid motion) of Euclidean space $\mathbb{R}^3$. The most important such properties are the curvature function κ and the torsion function τ. Once they have been defined one proves the *Fundamental Theorem of Space Curves*, which states that for any given continuous functions κ, τ there is a space curve with these curvature and torsion functions, and, that

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

this curve is uniquely determined up to a rigid motion.

To define curvature, observe that at each point of a parametrized space curve $c(t)$ there is a parametrized circle $\gamma(t)$ with

$$c(t_0) = \gamma(t_0), \dot{c}(t_0) = \dot{\gamma}(t_0), \ddot{c}(t_0) = \ddot{\gamma}(t_0).$$

This circle – which may degenerate to a straight line – is called the *osculating circle at* t_0, its radius is called *curvature radius at* t_0 and the inverse of the radius is called *the curvature at* t_0, $\kappa(t_0)$. The computation of curvature is simpler if the curve is parametrized by arc length, i.e. if the length of all tangent vectors is one, $|\dot{c}(t)| = 1$. One gets $\kappa(t) = |\ddot{c}(t)|$. Check this for the circle $c(t) := r \cdot (\cos(t/r),\ \sin(t/r))$. The most common way to proceed is to assume that $\kappa(t) > 0$. This allows one to define the *Frenet basis* along the curve:

$$\begin{aligned} e_1(t) &:= \dot{c}(t), \\ e_2(t) &:= \ddot{c}(t)/\kappa(t), \\ e_3(t) &:= e_1(t) \times e_2(t). \end{aligned}$$

The Frenet basis defines three curves $t \mapsto e_j(t)$ on the unit sphere. To emphasize the fact that $e_j(t)$ are to be considered as vectors, not as points, one calls the length of their derivative, $|\dot{e}_j(t)|$, *angular velocity* or *rotation speed* and not just velocity. For example, the formula $\ddot{c}(t) = \kappa(t)e_2(t)$ says that $\kappa(t)$ is the rotation speed of $\dot{c}(t)$. Next, we get from $\dot{e}_1(t) \sim e_2(t)$ that the derivative of $e_3(t)$ is proportional to $e_2(t)$. This proportionality factor, the rotation speed of $e_3(t)$, is called the torsion function $\tau(t)$ of the curve $c(t)$. In formulas: $\tau(t) := \langle \dot{e}_3(t),\ e_2(t)\rangle$.

Now one changes the point of view and considers the two functions κ, τ as given. This turns the equations that were originally definitions of κ and τ into differential equations, the famous

Frenet-Serret Equations:

$$\begin{aligned}
\dot{e}_1(t) &= \qquad\qquad \kappa(t)\cdot e_2(t),\\
\dot{e}_2(t) &= -\kappa(t)\cdot e_1(t) - \tau(t)\cdot e_3(t),\\
\dot{e}_3(t) &= \qquad\qquad \tau(t)\cdot e_2(t),
\end{aligned}$$

or, with $\vec{\omega}(t) := -\tau(t)\cdot e_1(t) + \kappa(t)\cdot e_3(t)$,

$$\dot{e}_j(t) = \vec{\omega}(t) \times e_j(t).$$

Finally $\quad \dot{c}(t) = e_1(t).$

For given continuous functions κ, τ these differential equations have — for given orthonormal initial values — unique orthonormal solutions $\{e_1(t), e_2(t), e_3(t)\}$. The curve $c(t) := \int^t e_1(s)ds$ is then parametrized by arc length and has the given curvature functions κ, τ.

The simplest curves in the plane, straight lines and circles, have constant curvature. One may wonder what constant curvature curves look like in $\mathbb{R}^3$. In 3DXM we illustrate the use of the Frenet-Serret equations by showing the following family of constant curvature curves:

$$\kappa(t) := aa,$$

$$\tau(t) := bb + cc \cdot \sin(t) + dd \cdot \sin(2t) + ee \cdot \sin(3t).$$

The function τ is, if $bb = 0$, skew symmetric at its zeros at 0 and π. This implies that the solution curves are symmetric with respect to the normal planes at these points. From this it follows that we can get *closed nonplanar curves of constant curvature* easily: the only requirement is that the angle of the normal planes at $c(0)$ and $c(\pi)$ has to be a rational multiple of π. *Every* $bb = 0$ *one-parameter family of examples in 3DXM therefore contains many closed examples* — select in the Animation Menu the default morph.

In the less symmetric case $bb \neq 0$ (but $dd = 0$) the function τ is even at the maxima and the minima, at $t = \pi/2$, $t = 3\pi/2$, and this implies that $180°$ rotation around e_2 at these points is also a symmetry of solution curves. This can be used to find more

closed curves by solving 2-parameter problems as follows: For every value of aa, bb use cc to make the distance between the normals 0. Now change aa or bb slowly (continuing to use cc for keeping the distance between the normals 0) and observe how the angle between the symmetry normals varies. If this angle hits a value $2k/n \cdot \pi$ then n copies of the computed piece fit together to a smoothly closed curve.
If one has selected 'Constant Curvature' in the Menu 'Space Curves' then there is in the Action Menu an entry 'Other Closed Curves'. It opens a submenu where one can select first $bb = 0$ examples which are also hit by the default morph. Then there are $bb \neq 0$ embedded examples, some of them knotted. Moreover, the 11-2-knot has nonvanishing torsion and strongly resembles a torus knot. This is no coincidence since one can find constant curvature curves on tori by solving a second order ODE, and it is again a 2-parameter problem to close these up. – The example 'like 6 helices' looks in another way as one would imagine constant curvature curves: made up of left winding and right winding pieces of helices.
Do not miss to select 'Show Osculating Circles & Evolute'. The constant radius of the osculating circles

shows the constant curvature and the rotating motion of the radius shows size and sign of the torsion.

In 3DXM one can choose in the Action Menu 'Parallel Frame'.This frame is designed to rotate as little as possible along the curve, in $\mathbb{R}^3$. This property is more obvious when one looks at the torus knots than at the constant curvature curves. For further details see curves of *constant torsion.* The main advantage of these parallel frames is that they neither make it neccessary to assume more than continuity of the second derivative $\ddot{c}$, nor that $\kappa > 0$ everywhere, even straight lines are not exceptional curves if one works with these frames. Their differential equation is also simple:

Frenet-Serret Equations for Parallel Frames:

$$\begin{aligned}
\dot{e}_1(t) &:= \qquad\quad a(t) \cdot e_2(t) + b(t) \cdot e_3(t), \\
\dot{e}_2(t) &:= -a(t) \cdot e_1(t), \\
\dot{e}_3(t) &:= -b(t) \cdot e_1(t).
\end{aligned}$$

With an antiderivative $T(t)$ of the torsion $\tau(t) = T'(t)$ we can of course write the twodimensional curvature vector $(a(t), b(t))$ in terms of $\kappa(t), \tau(t)$, namely:

$$(a(t), b(t)) := \kappa(t)\big(\cos(T(t)), \sin(T(t))\big).$$