Almost Ramsey cardinals
IMST 2007

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Bonn, May 17, 2007

A set $X \subseteq \delta$ is *homogeneous* for a *partition* $F\colon [\delta]^{<\omega} \rightarrow 2$ iff $\forall n |F[[X]^n]| = 1$;

the *partition property* $\delta \rightarrow (\alpha)_2^{<\omega}$ is defined as

$$\forall F\colon [\delta]^{<\omega} \rightarrow 2 \exists X \subseteq \kappa (\mathrm{otp}(X) \geqslant \alpha \wedge X \text{is homogeneous for } F).$$

An infinite cardinal κ is α-Erdös iff $\kappa \rightarrow (\alpha)_2^{<\omega}$,

it is Ramsey iff $\kappa \rightarrow (\kappa)_2^{<\omega}$.

Definition 1. *An infinite cardinal* κ *is* almost Ramsey *iff* $\forall \alpha < \kappa\, \kappa \rightarrow (\alpha)_2^{<\omega}$.

For any uncountable almost RAMSEY cardinal κ the following substructure property holds: if λ, κ', λ' are infinite cardinals satisfying $\lambda \leqslant \kappa$, $\lambda' \leqslant \kappa' < \kappa$, and $\lambda' \leqslant \lambda$ then $(\kappa, \lambda) \Rightarrow (\kappa', \lambda')$, which means that every first-order structure $(\kappa, \lambda, \ldots)$ with a countable language has an elementary substructure $X \prec (\kappa, \lambda, \ldots)$ with $|X| = \kappa'$ and $|X \cap \lambda| = \lambda'$.

Theorem 2. *Con(ZFC + There exist cardinals $\kappa < \lambda$ such that κ is 2^λ supercompact where λ is the least regular almost* RAMSEY *cardinal greater than κ) implies Con(ZF + $\neg \mathrm{AC}_\omega$ + Every successor cardinal is regular + Every (well-ordered) uncountable cardinal is almost* RAMSEY*).*

Theorem 3. *Assume* ZF *and that every infinite cardinal is almost* RAMSEY*. Then there exists an inner model with a strong cardinal.*

Theorem 4. *The following theories are equiconsistent*

a) ZFC + *There is a proper class of regular almost* RAMSEY *cardinals;*

b) ZF + *All infinite cardinals except possibly successors of singular limit cardinals are almost* RAMSEY*.*

Definition 5. *For* $\alpha \in \mathrm{Ord}$ *let* $\kappa(\alpha)$ *be the least* κ *such that* $\kappa \to (\alpha)^{<\omega}_2$, *if such a* κ *exists.*

Proposition 6. (ZF) *An infinite cardinal* κ *is almost* Ramsey *iff* $\kappa(\alpha)$ *is defined for all* $\alpha < \kappa$ *and* $\kappa = \bigcup_{\alpha<\kappa} \kappa(\alpha)$.

Proposition 7. (ZFC) *Assume* κ *is almost* Ramsey*. Then*

a) $\forall \alpha < \kappa\ \kappa(\alpha) < \kappa$;

b) κ *is a strong limit cardinal.*

Proposition 8. *Let* M *be a transitive model of "*ZFC + κ *is almost* Ramsey*". Let* $N \supseteq M$ *be a transitive model of* ZFC *such that* $\forall \delta < \kappa\ \mathcal{P}(\delta) \cap M = \mathcal{P}(\delta) \cap N$. *Then* κ *is almost* Ramsey *in* N.

Proof. Let $\alpha < \kappa$. By Proposition 7, $\kappa(\alpha)^M < \kappa$. $\mathcal{P}(\kappa(\alpha)^M) \cap M = \mathcal{P}(\kappa(\alpha)^M) \cap N$ implies that $\kappa(\alpha)^N = \kappa(\alpha)^M$. Hence $\kappa = \bigcup_{\alpha<\kappa} \kappa(\alpha)^N$ and κ is almost Ramsey in N. □

Proposition 9. (ZFC)

a) Assume λ *is a* Ramsey *cardinal. Then the class of almost* Ramsey *cardinals is closed unbounded below* λ *and the class of* regular *almost* Ramsey *cardinals is stationary below* λ.

b) Assume κ *is an uncountable* regular *almost* Ramsey *cardinal. Then the class of almost* Ramsey *cardinals is closed unbounded below* λ.

Proposition 10. ZFC + *There exists an uncountable regular almost* Ramsey *cardinal* $\vdash$ Con(ZFC + *There exists a proper class of (singular) almost* Ramsey *cardinal.*

Proposition 11. (ZF) *For infinite ordinals α the partition property $\kappa \rightarrow (\alpha)_2^{<\omega}$ is equivalent to: for any first-order structure $\mathcal{M} = (M, \ldots)$ in a countable language S with $\kappa \subseteq M$ there is a set $X \subseteq \kappa$,* $\operatorname{otp}(X) \geqslant \alpha$ *of* indiscernibles*, i.e., for all S-formulas $\varphi(v_0, \ldots, v_{n-1})$, $x_0, \ldots, x_{n-1} \in X$, $x_0 < \ldots < x_{n-1}$, $y_0, \ldots, y_{n-1} \in X$, $y_0 < \ldots < y_{n-1}$ holds*

$$\mathcal{M} \vDash \varphi(x_0, \ldots, x_{n-1}) \text{ iff } \mathcal{M} \vDash \varphi(y_0, \ldots, y_{n-1}).$$

Proposition 12. (ZF) *Assume $\kappa \rightarrow (\alpha)_2^{<\omega}$ where α is a limit ordinal. Then for any first-order structure $\mathcal{M} = (M, \ldots)$ in a countable language S with $\kappa \subseteq M$ there is a set $X \subseteq \kappa$,* $\operatorname{otp}(X) \geqslant \alpha$ *of* good indiscernibles*, i.e., for all S-formulas $\varphi(v_0, \ldots, v_{m-1}, w_0, \ldots, w_{n-1})$, $x_0, \ldots, x_{n-1} \in X$, $x_0 < \ldots < x_{n-1}$, $y_0, \ldots, y_{n-1} \in X$, $y_0 < \ldots < y_{n-1}$, and $a_0 < \ldots < a_{m-1} < \min(x_0, y_0)$ holds*

$$\mathcal{M} \vDash \varphi(a_0, \ldots, a_{m-1}, x_0, \ldots, x_{n-1}) \text{ iff } \mathcal{M} \vDash \varphi(a_0, \ldots, a_{m-1}, y_0, \ldots, y_{n-1}).$$

Proof. We may assume that the structure $\mathcal{M}$ contains a unary predicate Ord for the ordinals in M ($=\kappa$) and a collection of SKOLEM functions for ordinal-valued existential statements, i.e., for every S-formula $\varphi(v,\vec{w})$ there is a function f of M such that

$$M\vDash\forall\vec{w}\,(\exists v(\mathrm{Ord}(v)\wedge\varphi(v,\vec{w}))\rightarrow\varphi(f(\vec{w}),\vec{w})).$$

Choose a set $X\subseteq\kappa$, $\mathrm{otp}(X)\geqslant\alpha$ of indiscernibles for M such that the minimum $\min\ (X)$ is minimal for all such sets of indiscernibles. Assume for a contradiction that X is not good. Then there is an S-formula $\varphi(v_0,\ldots,v_{n-1})$, $x_0,\ldots,x_{n-1}\in X$, $x_0<\ldots<x_{n-1}$, $y_0,\ldots,y_{n-1}\in X$, $y_0<\ldots<y_{n-1}$ and $a_0<\ldots<a_{m-1}<\min\,(x_0,y_0)$ such that

$$M\vDash\varphi(a_0,\ldots,a_{m-1},x_0,\ldots,x_{n-1})\text{ and }M\vDash\neg\varphi(a_0,\ldots,a_{m-1},y_0,\ldots,y_{n-1}).$$

Since α is a limit ordinal we can take $z_0,\ldots,z_{n-1}\in X$, $z_0<\ldots<z_{n-1}$ such that $x_{n-1}<z_0$ and $y_{n-1}<z_0$. In case $M\vDash\varphi(a_0,\ldots,a_{m-1},z_0,\ldots,z_{n-1})$, one has

$$M\vDash\neg\varphi(a_0,\ldots,a_{m-1},y_0,\ldots,y_{n-1})\text{and }M\vDash\varphi(a_0,\ldots,a_{m-1},z_0,\ldots,z_{n-1})$$

where $y_0<\ldots<y_{n-1}<z_0<\ldots<z_{n-1}$.

In case $M\vDash\neg\varphi(a_0,\ldots,a_{m-1},z_0,\ldots,z_{n-1})$, one has

$$M\vDash\varphi(a_0,\ldots,a_{m-1},x_0,\ldots,x_{n-1})\text{ and }M\vDash\neg\varphi(a_0,\ldots,a_{m-1},z_0,\ldots,z_{n-1})$$

where $x_0<\ldots<x_{n-1}<z_0<\ldots<z_{n-1}$. So in both cases we have an ascending $2n$-tuble of indiscernibles, such that the first half behaves differently from the second half with respect to the formula φ and the parameters $a_0,\ldots,a_{m-1}$. So without loss of generality we may assume that $x_0<\ldots<x_{n-1}<y_0<\ldots<y_{n-1}$ and

$$M\vDash\varphi(a_0,\ldots,a_{m-1},x_0,\ldots,x_{n-1})\text{ and }M\vDash\neg\varphi(a_0,\ldots,a_{m-1},y_0,\ldots,y_{n-1}).$$

Write $\vec{x} = x_0, \ldots, x_{n-1}$ and $\vec{y} = y_0, \ldots, y_{n-1}$. Since M contains Skolem functions there are functions $f_0, \ldots, f_{m-1}$ of M which compute parameters like $a_0, \ldots, a_{m-1}$:

$M \vDash \exists v_1 < x_0 \exists v_2 < x_1 \ldots \exists v_{m-1} < x_0 (f_0(\vec{x}, \vec{y}) < x_0 \wedge \varphi(f_0(\vec{x}, \vec{y}), v_1, \ldots, v_{m-1}, \vec{x}) \wedge \neg\varphi(f_0(\vec{x}, \vec{y}), v_1, \ldots, v_{m-1}, \vec{y}))$

$M \vDash \exists v_2 < x_0 \ldots \exists v_{m-1} < x_0 (f_0(\vec{x}, \vec{y}) < x_0 \wedge f_1(\vec{x}, \vec{y}) < x_0 \wedge \varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, v_{m-1}, \vec{x}) \wedge \neg\varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, v_{m-1}, \vec{y}))$

$\vdots$

$M \vDash f_0(\vec{x}, \vec{y}) < x_0 \wedge \ldots \wedge f_{m-1}(\vec{x}, \vec{y}) < x_0 \wedge \varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, f_{m-1}(\vec{x}, \vec{y}), \vec{x}) \wedge \neg\varphi(f_0(\vec{x}, \vec{y}), f_1(\vec{x}, \vec{y}), \ldots, f_{m-1}(\vec{x}, \vec{y}), \vec{y})$

Now consider $\vec{z} = z_0, \ldots, z_{n-1} \in X$, $z_0 < \ldots < z_{n-1}$ such that $y_{n-1} < z_0$.

(1) There is $k < m$ such that $f_k(\vec{x}, \vec{y}) \neq f_k(\vec{y}, \vec{z})$.

Proof. Assume not. Set $\xi_0 = f_0(\vec{x}, \vec{y}), \ldots, \xi_{m-1} = f_{m-1}(\vec{x}, \vec{y})$. Then

$$M \ \vDash \ \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{x}) \wedge \neg\varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{y})$$

and

$$M \ \vDash \ \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{y}) \wedge \neg\varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{z}).$$

In particular

$$M \ \vDash \ \varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{y}) \wedge \neg\varphi(\xi_0, \xi_1, \ldots, \xi_{m-1}, \vec{y}),$$

which is a contradiction. □

So take $k < m$ such that

(2) $f_k(\vec{x}, \vec{y}) \neq f_k(\vec{y}, \vec{z})$.

Let $(\nu_i | i < \alpha)$ be a strictly increasing enumeration of the set X of indiscernibles, and let $(\vec{x}^{(i)} | i < \alpha)$ with

$$\vec{x}^{(i)} = \nu_{n \cdot i}, \nu_{n \cdot i + 1}, \ldots, \nu_{n \cdot i + n - 1}$$

be a partition of X into ascending sequences of length n.

(3) $f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) < f_k(\vec{x}^{(1)}, \vec{x}^{(2)})$.

Proof. By indiscernibility, (2) implies that $f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) \neq f_k(\vec{x}^{(1)}, \vec{x}^{(2)})$. Assume for a contradiction that $f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) > f_k(\vec{x}^{(1)}, \vec{x}^{(2)})$. Then again by indiscernibility we would obtain a *decreasing* $\in$ -sequence

$$f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) > f_k(\vec{x}^{(1)}, \vec{x}^{(2)}) > f_k(\vec{x}^{(2)}, \vec{x}^{(3)}) > \ldots,$$

contradiction. $\square$

But then

$$f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) < f_k(\vec{x}^{(2)}, \vec{x}^{(3)}) < f_k(\vec{x}^{(4)}, \vec{x}^{(5)}) < \ldots$$

is an *ascending* α-sequence of indiscernibles for M with smallest element $f_k(\vec{x}^{(0)}, \vec{x}^{(1)}) < \nu_0$ which contradicts the minimal choice of min (X). $\square$

Lemma 13. (ZF) *Let* κ^+ *be almost* RAMSEY. *Then* $(\kappa^+)^{\mathrm{HOD}} < \kappa^+$.

Proof. Assume for a contradiction that $(\kappa^+)^{\mathrm{HOD}} = \kappa^+$. For $\gamma \in [\kappa, \kappa^+)$ choose the $<_{\mathrm{HOD}}$-least bijection $f_\gamma: \gamma \leftrightarrow \kappa$. Define $F: [\kappa]^3 \rightarrow 2$ by

$$F(\{\alpha, \beta, \gamma\}) = \begin{cases} 0 \text{ iff } f_\gamma(\alpha) < f_\gamma(\beta) \\ 1 \text{ iff } f_\gamma(\alpha) > f_\gamma(\beta) \end{cases}, \text{ for } \alpha < \beta < \gamma.$$

Take $X \subseteq \kappa^+$ homogeneous for F with $\mathrm{otp}(X) = \kappa + 2$. Let $\gamma = \max(X)$. Then define $h: \kappa + 1 \to \kappa$ by $h(\xi) = f_\gamma(\alpha_\xi)$ where α_ξ is the ξ-th element of X. *Case* 1: $\forall x \in [X]^3 F(x) = 0$. Then for $\xi < \zeta < \kappa + 1$ we have: $\alpha_\xi < \alpha_\zeta < \gamma$, $\{\alpha_\xi, \alpha_\zeta, \gamma\} \in [X]^3$, $F(\{\alpha_\xi, \alpha_\zeta, \gamma\}) = 0$, and so

$$h(\xi) = f_\gamma(\alpha_\xi) < f_\gamma(\alpha_\zeta) = h(\zeta).$$

Thus $h: \kappa + 1 \to \kappa$ is order preserving, which is impossible.

Case 2: $\forall x \in [X]^3 F(x) = 1$. Then for $\xi < \zeta < \kappa + 1$ we have: $\alpha_\xi < \alpha_\zeta < \gamma$, $\{\alpha_\xi, \alpha_\zeta, \gamma\} \in [X]^3$, $F(\{\alpha_\xi, \alpha_\zeta, \gamma\}) = 1$, and so

$$h(\xi) = f_\gamma(\alpha_\xi) > f_\gamma(\alpha_\zeta) = h(\zeta).$$

Thus $h: \kappa + 1 \to \kappa$ is a strictly descending $\kappa + 1$ chain in the ordinals, contradiction. □

Let K^{DJ} be the canonical term for the DODD-JENSEN core model.

Proposition 14. (ZF) *Let* $a \subseteq \mathrm{HOD}$ *be a set. Then*

a) $\mathrm{HOD}[a]$ *is a set-generic extension of* HOD, *so* $\mathrm{HOD}[a] \vDash \mathrm{ZFC}$.

b) $(K^{\mathrm{DJ}})^{\mathrm{HOD}} = (K^{\mathrm{DJ}})^{\mathrm{HOD}[a]}$*; moreover this equality holds for every level of the hierarchy, i.e.,* $(K^{\mathrm{DJ}}_\alpha)^{\mathrm{HOD}} = (K^{\mathrm{DJ}}_\alpha)^{\mathrm{HOD}[a]}$ *for every* $\alpha \in \mathrm{Ord}$.

By the proposition we may define $K^{\mathrm{DJ}} = (K^{\mathrm{DJ}})^{\mathrm{HOD}}$ in models without choice.

Proposition 15. *Let* κ *be an infinite cardinal and suppose* $A \in K^{\mathrm{DJ}} \cap \mathcal{P}(K^{\mathrm{DJ}}_\kappa)$*, and that there is* I*, an infinite good set of indiscernibles for* $\mathcal{A} = (K^{\mathrm{DJ}}_\kappa, A)$ *and that* $\mathrm{cof}(\mathrm{otp}(I)) > \omega$. *Then there is* $I' \in K^{\mathrm{DJ}}$, $I' \supseteq I$ *a set of good indiscernibles for* $\mathcal{A}$.

Lemma 16. (ZF) *Let $\kappa > \aleph_1$ be almost* RAMSEY. *Then κ is almost* RAMSEY *in K^{DJ}.*

Proof. Let $F\colon [\kappa]^{<\omega} \to 2$, $F \in K^{\mathrm{DJ}}$ be a partition. Let $\alpha < \kappa$. Then $\alpha + \aleph_1 < \kappa$. By Proposition 12, Take a set $X \subseteq \kappa$ of *good* indiscernibles for the structure $M = (K^{\mathrm{DJ}}_\kappa, F)$ with $\mathrm{otp}(X) \geqslant \alpha + \aleph_1$. Let X' be the initial segment of X of order type $(\alpha + \aleph_1)^{\mathrm{HOD}(X)}$. In the model $\mathrm{HOD}(X)$, X' is a good set of indiscernibles for M such that $\mathrm{cof}(\mathrm{otp}(X')) > \omega$. By the indiscernibles lemma applied inside $\mathrm{HOD}(X)$ there is a set $Y \supseteq X'$, $Y \in K$ which is a good set of indiscernibles for M. Then Y is also homogeneous for the partition F of ordertype $\geqslant \alpha$. $\square$

We are now able to prove the inner model direction of Theorem 4:

Lemma 17. Con(ZF + *All infinite cardinals except possibly successors of singular limit cardinals are almost* RAMSEY) *implies* Con(ZFC + *There is a proper class of regular almost* RAMSEY *cardinals*).

Proof. Assume Con(ZF + All infinite cardinals except possibly successors of singular limit cardinals are almost RAMSEY). If there is a proper class of *regular* almost RAMSEY cardinals, we are done. So assume that this is not the case, and let the cardinal θ be an upper bound for the set of regular almost RAMSEY cardinals. Then θ^{++} and θ^{+++} are not successors of limit cardinals. By assumption, θ^{++} and θ^{+++} are almost RAMSEY. By the definition of θ, θ^{++} and θ^{+++} must be singular. By [Sc99], this implies consistency strength far above RAMSEY cardinals. $\square$

In the following we apply the core model below a strong cardinal, denoted by the class term K. As for the DODD-JENSEN core model we get:

Proposition 18. (ZF) *Let* $a \subseteq \mathrm{HOD}$ *be a set. Then* $K^{\mathrm{HOD}} = K^{\mathrm{HOD}[a]}$.

If there is no inner model with a strong cardinal and the axiom of choice holds then the core model K satisfies the weak covering theorem, i.e., for sufficiently large singular cardinals κ we have $\kappa^+ = (\kappa^+)^K$.

Lemma 19. (ZF) *Let κ^+ be almost* Ramsey *where κ is a singular cardinal $\geqslant \aleph_2$. Then there is an inner model with a strong cardinal.*

Proof. Assume that there is no inner model with a strong cardinal. By Lemma 13, $(\kappa^+)^{\mathrm{HOD}} < \kappa^+$. Since $K \subseteq \mathrm{HOD}$, $(\kappa^+)^K < \kappa^+$. Choose a bijection $f\colon \kappa \leftrightarrow (\kappa^+)^K$ and a cofinal subset $Z \subseteq \kappa$ such that $\mathrm{otp}(Z) < \kappa$. The class $\mathrm{HOD}(f, Z)$ is a model of ZFC and it satisfies that κ is a singular cardinal such that $(\kappa^+)^K < \kappa^+$. But this contradicts the covering theorem below 0^{pistol} inside the model $\mathrm{HOD}(f, Z)$. □

Lemma 20. *Assume* ZF *and that every infinite cardinal is almost* Ramsey*. Then there exists an inner model with a strong cardinal.*

Proof. By assumption, $\aleph_{\omega+1}$ is almost Ramsey and the successor of the singular cardinal $\aleph_\omega \geqslant \aleph_2$. Now use Lemma 19. □