It is relatively consistent that the infinite cardinals are alternately regular and singular

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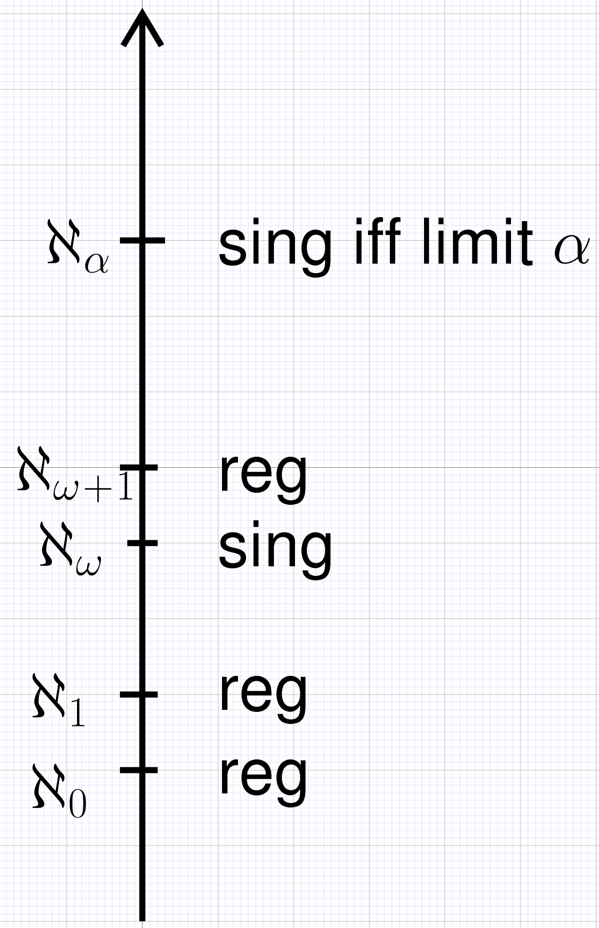
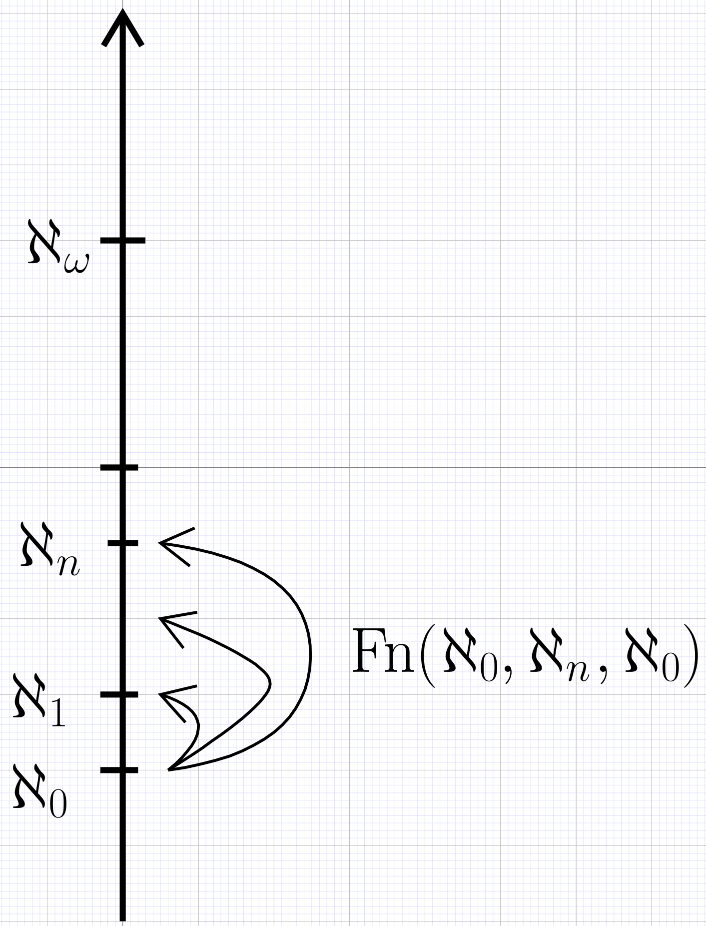
ZFC



$\aleph_\alpha$
sing iff limit α
$\aleph_{\omega+1}$
reg
$\aleph_\omega$
sing
$\aleph_1$
reg
$\aleph_0$
reg

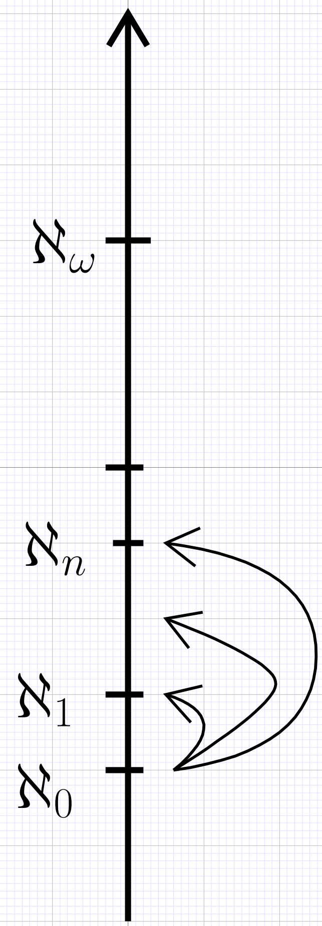
$\mathrm{ZF} + \neg\mathrm{AC}$

The Feferman-Levy model



$\mathrm{ZF} + \neg\mathrm{AC}$

The Feferman-Levy model

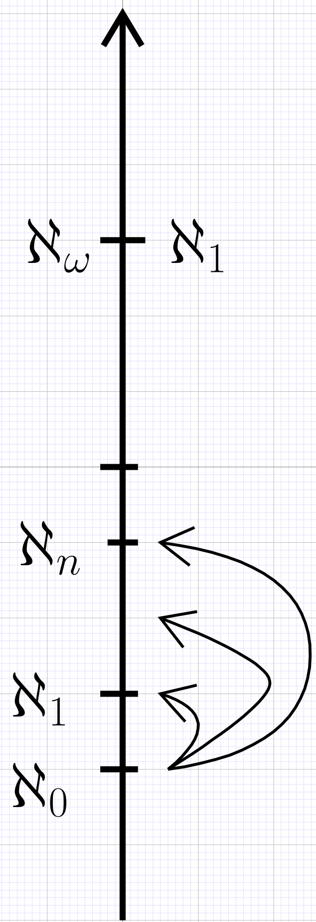


$\prod^{\mathrm{fin}}_{n<\omega} \mathrm{Fn}(\aleph_0, \aleph_n, \aleph_0)$

$\mathrm{Fn}(\aleph_0, \aleph_n, \aleph_0)$

$\mathrm{ZF} + \neg\mathrm{AC}$

The Feferman-Levy model



symmetric submodel

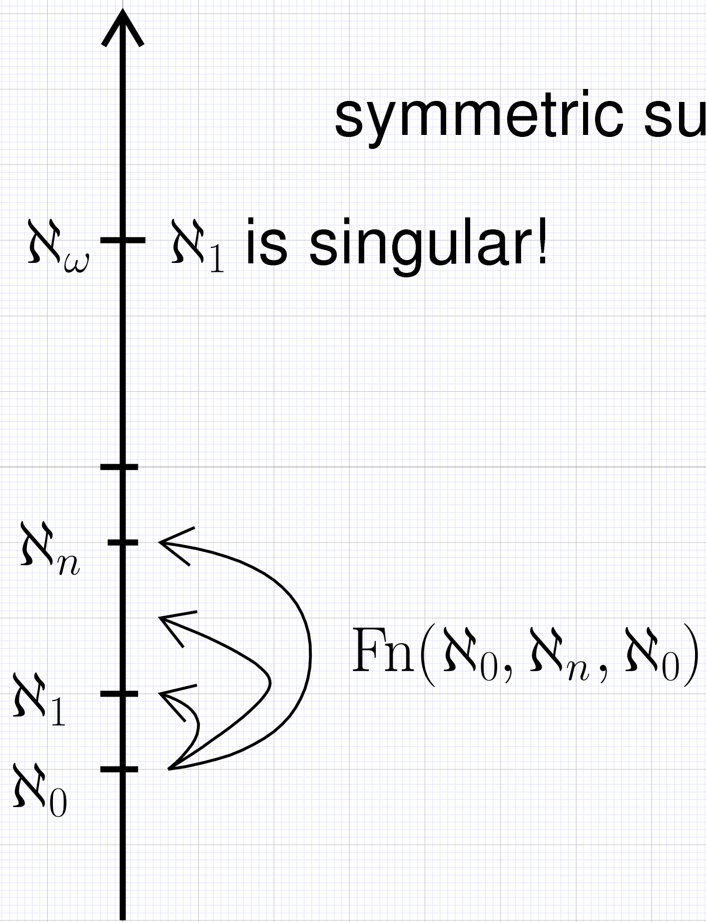
$$\prod_{n<\omega}^{\mathrm{fin}} \mathrm{Fn}(\aleph_0, \aleph_n, \aleph_0)$$

$$\mathrm{Fn}(\aleph_0, \aleph_n, \aleph_0)$$

$\mathrm{ZF} + \neg\mathrm{AC}$

symmetric submodel

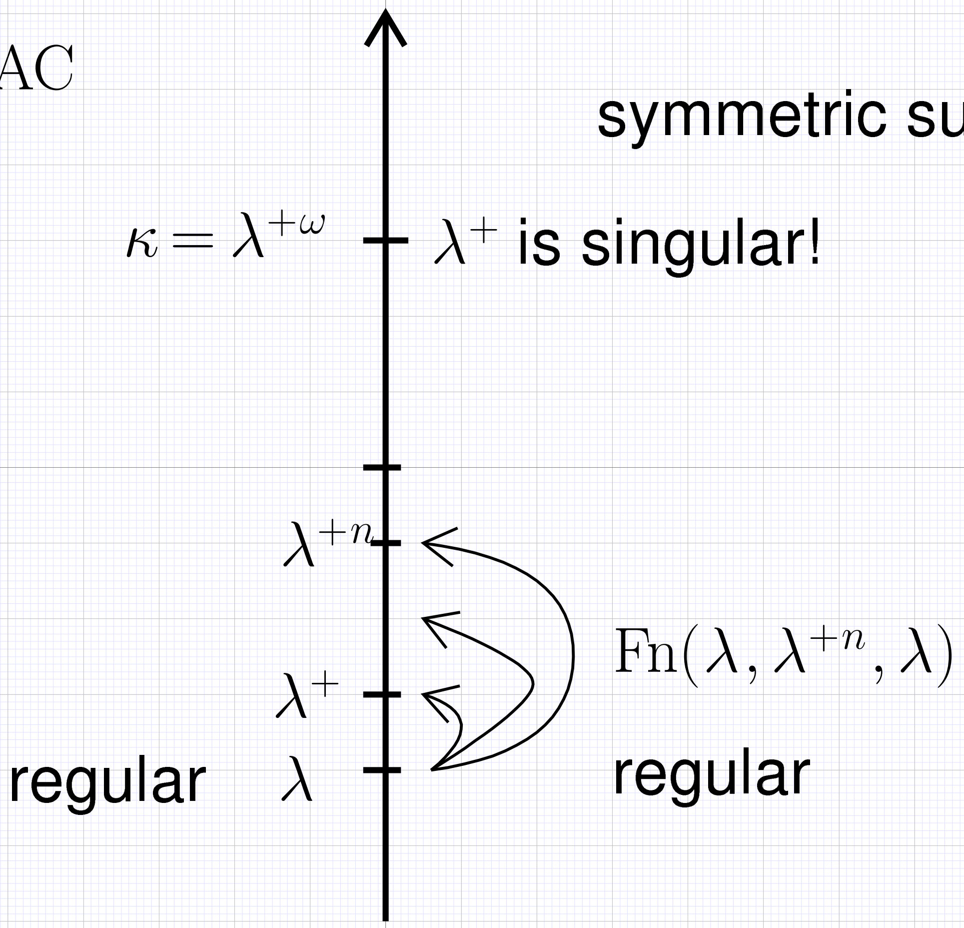
The Feferman-Levy model

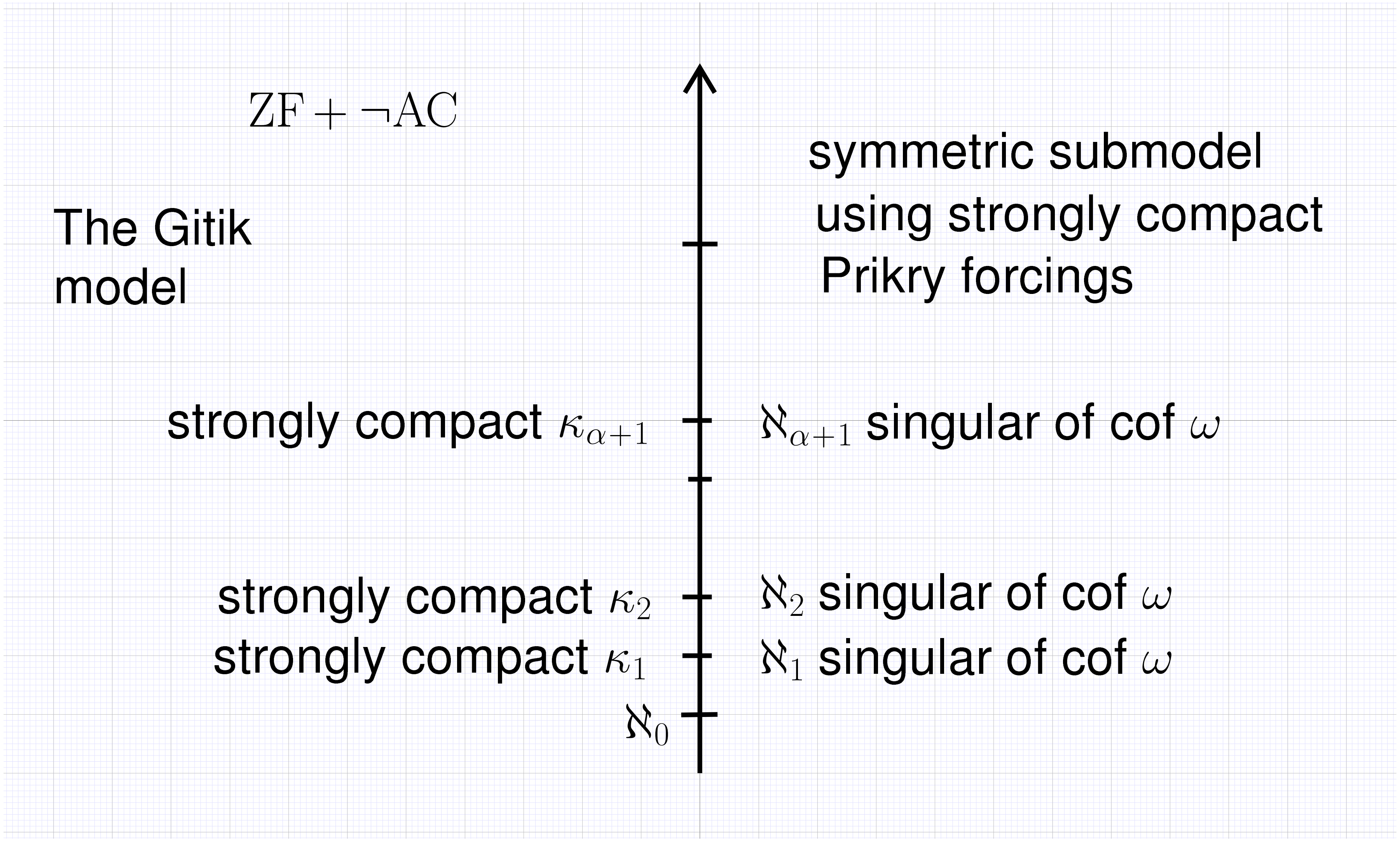


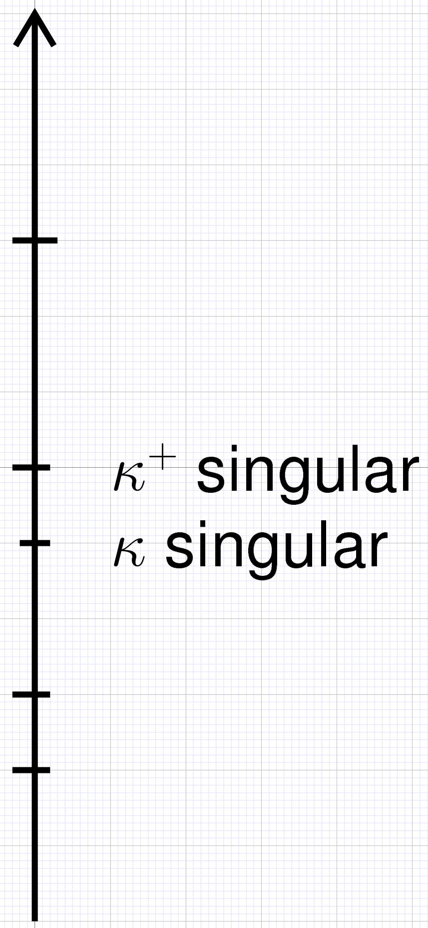
$\mathrm{ZF} + \neg\mathrm{AC}$

symmetric submodel

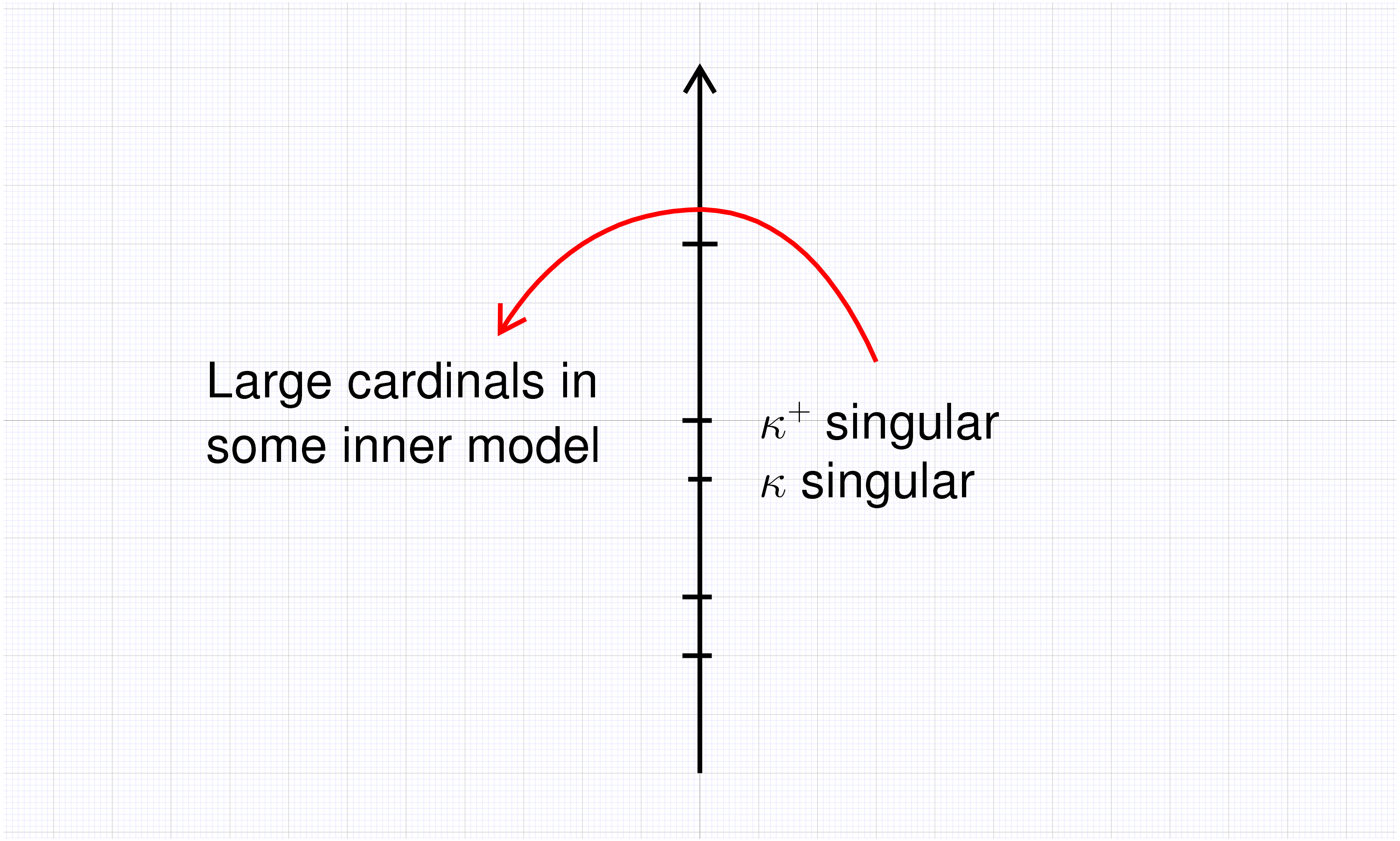
The Feferman-Levy model (variant)

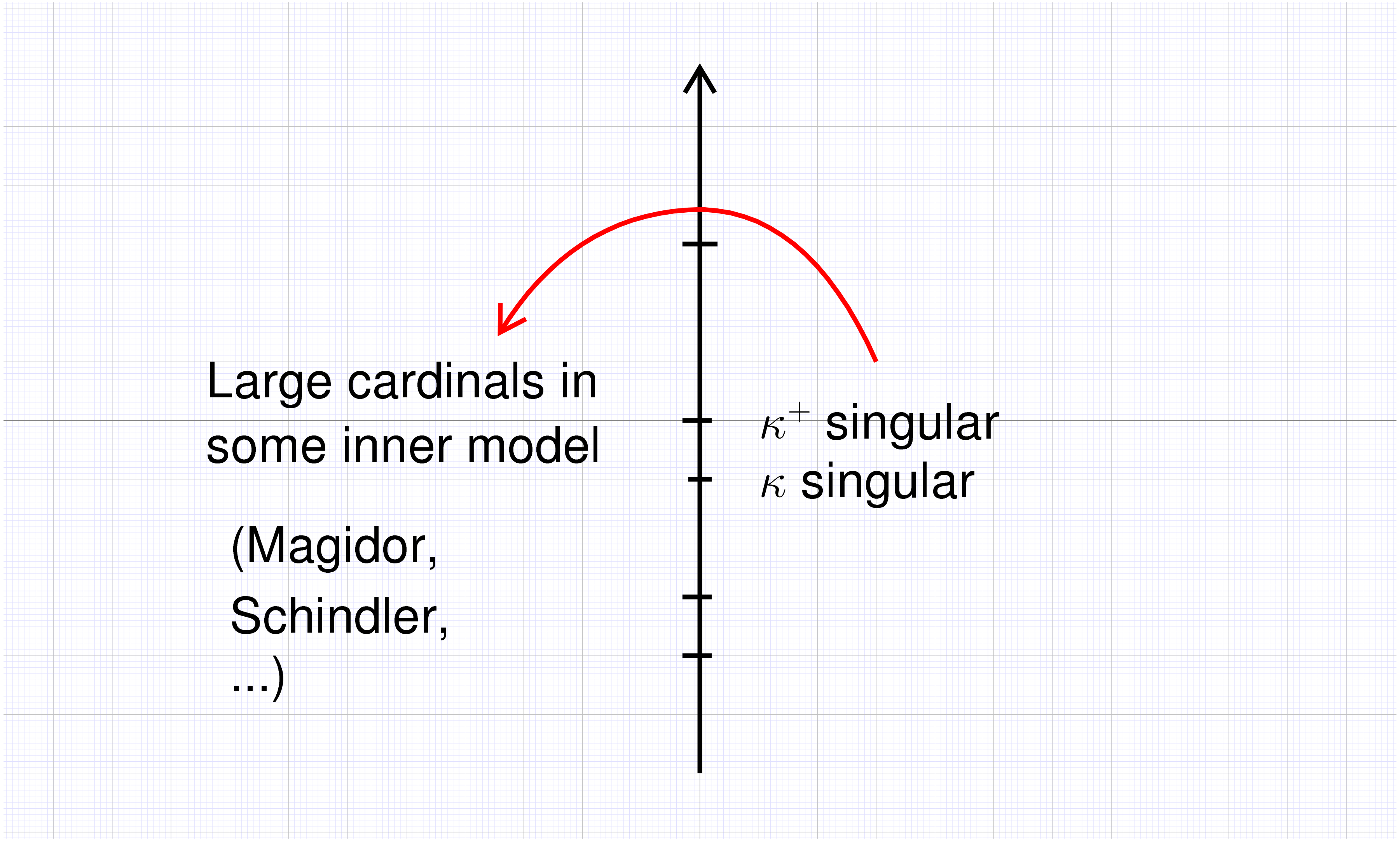



$\mathrm{ZF} + \neg\mathrm{AC}$
The Gitik model
symmetric submodel using strongly compact Prikry forcings
strongly compact $\kappa_{\alpha+1}$
$\aleph_{\alpha+1}$ singular of cof ω
strongly compact κ_2
$\aleph_2$ singular of cof ω
strongly compact κ_1
$\aleph_1$ singular of cof ω
$\aleph_0$

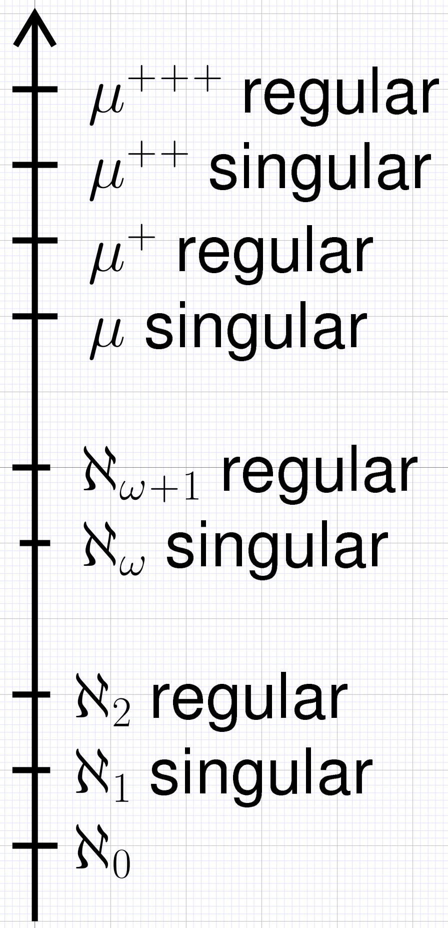


κ^+ singular
κ singular

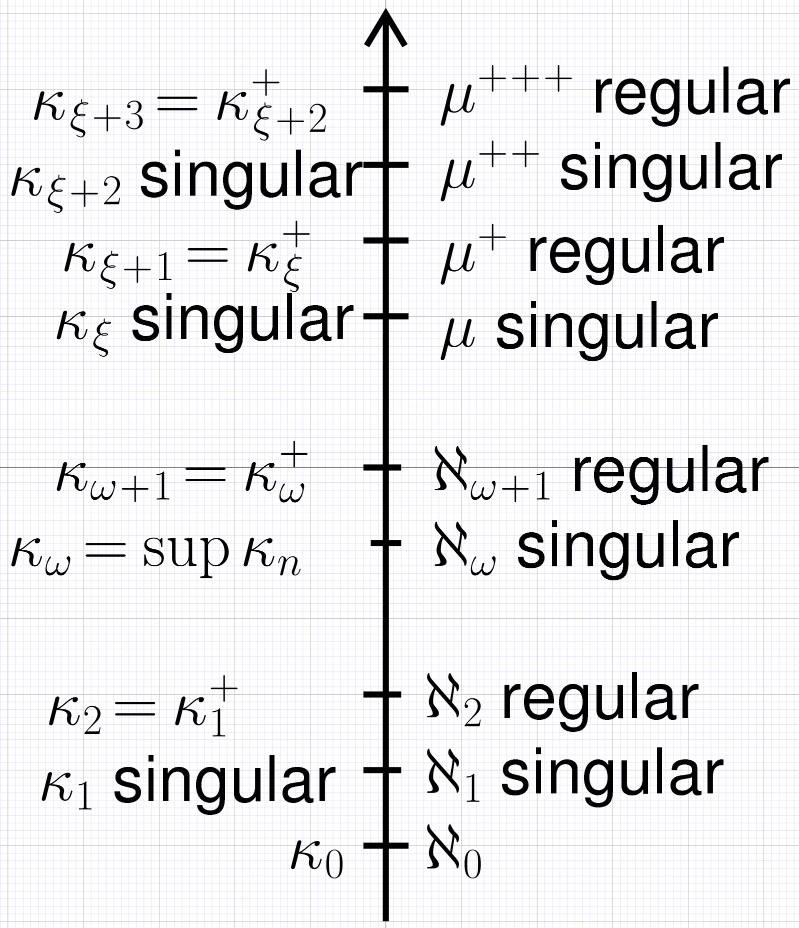

Large cardinals in
some inner model
κ^+ singular
κ singular


Large cardinals in some inner model
(Magidor, Schindler, ...)
κ^+ singular
κ singular

Aim:

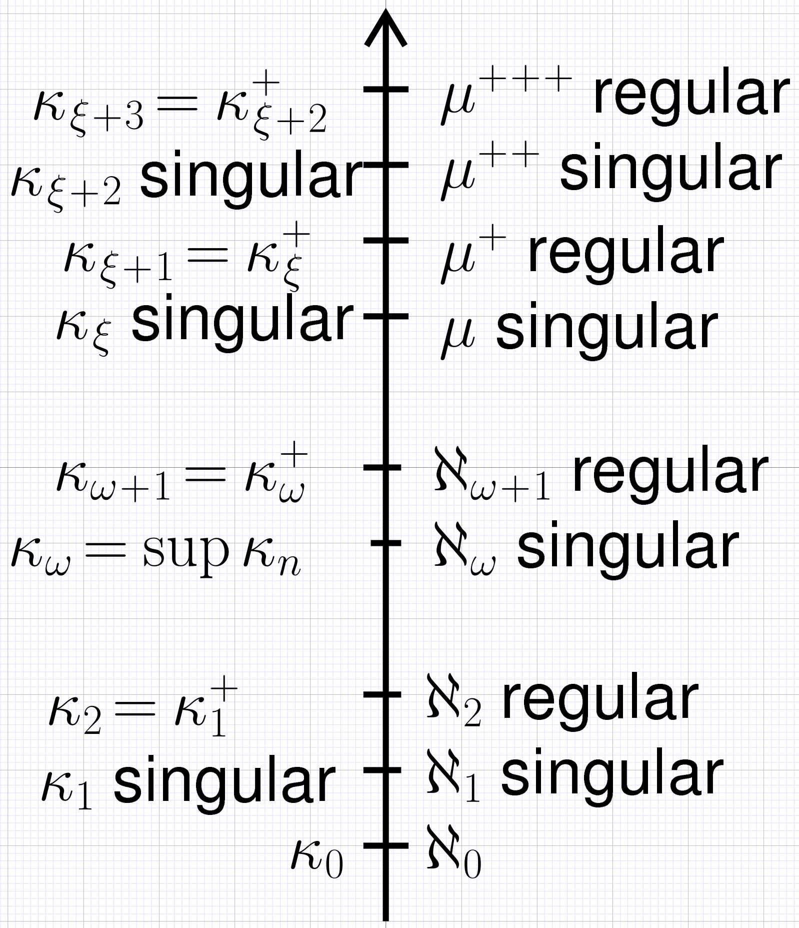

μ^{+++} regular
μ^{++} singular
μ^{+} regular
μ singular
$\aleph_{\omega+1}$ regular
$\aleph_{\omega}$ singular
$\aleph_2$ regular
$\aleph_1$ singular
$\aleph_0$

In ground model of ZFC+GCH choose



In ground model of ZFC+GCH choose

Then do finite support product of Feferman-Levy



Details of the forcing

$$P = \prod_{\kappa_i \text{ regular}, \kappa_i < \nu < \kappa_{i+1}}^{\text{fin}} \mathrm{Fn}(\kappa_i, \nu, \kappa_i)$$

$$G = \prod_{\kappa_i \text{ regular}, \kappa_i < \nu < \kappa_{i+1}}^{\text{fin}} G_\nu$$

$$g_\nu := \bigcup G_\nu \colon \kappa_i \twoheadrightarrow \nu \text{ surjective}$$

Fuzzyfying the sequence of g_ν's

$$g_\nu := \bigcup G_\nu \colon \kappa_i \twoheadrightarrow \nu \text{ surjective}$$

$$\tilde{g}_\nu := \{ f \colon \kappa_i \twoheadrightarrow \nu \mid \exists p, q \in \mathrm{Fn}(\kappa_i, \nu, \kappa_i) \colon f = (g_n \setminus p) \cup q \}$$

$$S = \bigcup_{\kappa_i \text{ regular}, \kappa_i < \nu < \kappa_{i+1}} \tilde{g}_\nu$$

$$N = \mathrm{HOD}^{V[G]}(S \cup \{S\})$$

The ‘‘symmetric’’ HOD model

$$N = \mathrm{HOD}^{V[G]}(S \cup \{S\}) \vDash \mathrm{ZF}$$

Every $X \in N$ is of the form

$$X = t(\vec{z}\,, g_{\nu_0}, \ldots, g_{\nu_{n-1}}, S)$$

where t is an abstraction term, $\vec{z} \in V$, $\nu_0 < \ldots < \nu_{n-1}$.

Weak homogeneity

Let $\varphi(\check{z}_0, \ldots, \check{z}_{m-1}, \dot{g}_{\nu_0}, \ldots, \dot{g}_{\nu_{n-1}}, \dot{S})$ be a forcing sentence with canonical names. Let $p, q \in P$ with

$$p \upharpoonright \{\nu_0, \ldots, \nu_{n-1}\} \| q \upharpoonright \{\nu_0, \ldots, \nu_{n-1}\}.$$

Then it is **not** possible that

$$p \Vdash \varphi(\check{z}_0, \ldots, \check{z}_{m-1}, \dot{g}_{\nu_0}, \ldots, \dot{g}_{\nu_{n-1}}, \dot{S})$$

and

$$q \Vdash \neg\varphi(\check{z}_0, \ldots, \check{z}_{m-1}, \dot{g}_{\nu_0}, \ldots, \dot{g}_{\nu_{n-1}}, \dot{S}).$$

Approximation property

Let $X = t(\vec{z}\,, g_{\nu_0}, \ldots, g_{\nu_{n-1}}, S) \subseteq \mathrm{Ord}$. Then $X \in V[G_{\nu_0} \times \ldots \times G_{\nu_{n-1}}]$.

Approximation property

Let $X = t(\vec{z}\,, g_{\nu_0}, \ldots, g_{\nu_{n-1}}, S) \subseteq \mathrm{Ord}$. Then $X = \{\xi \,|\forall\, r \in G_{\nu_0} \times \ldots \times G_{\nu_{n-1}}\, \exists p \in P\, (p \upharpoonright \{\nu_0, \ldots, \nu_{n-1}\} \supseteq r$ and $p \Vdash \check{\xi} \in t(\check{\vec{z}}\,, \dot{g}_{\nu_0}, \ldots, \dot{g}_{\nu_{n-1}}, \dot{S}))\} \in V[G_{\nu_0} \times \ldots \times G_{\nu_{n-1}}]$.

Approximation property

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Proof. $\subseteq \checkmark$.

Approximation property

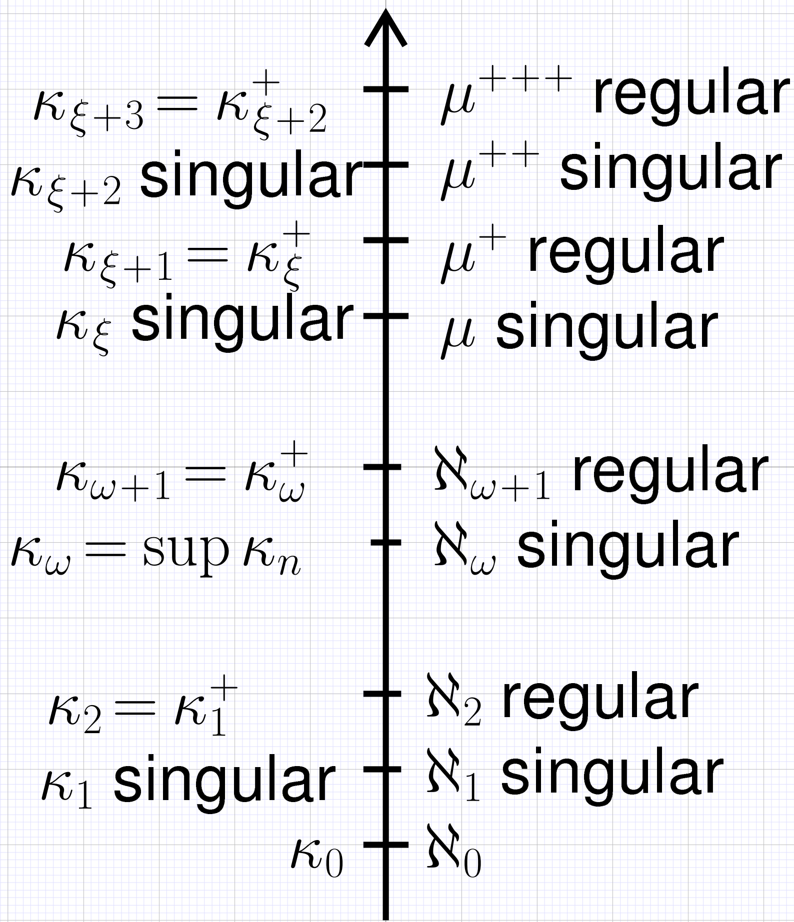
Let $X = t(\vec{z}\,, g_{\nu_0}, \ldots, g_{\nu_{n-1}}, S) \subseteq \mathrm{Ord}$. Then $X = \{\xi \,|\forall\, r \in G_{\nu_0} \times \ldots \times G_{\nu_{n-1}}\, \exists p \in P\,(p \restriction \{\nu_0, \ldots, \nu_{n-1}\} \supseteq r$ and $p \Vdash \check{\xi} \in t(\check{\vec{z}}\,, \dot{g}_{\nu_0}, \ldots, \dot{g}_{\nu_{n-1}}, \dot{S}))\} \in V[G_{\nu_0} \times \ldots \times G_{\nu_{n-1}}]$.

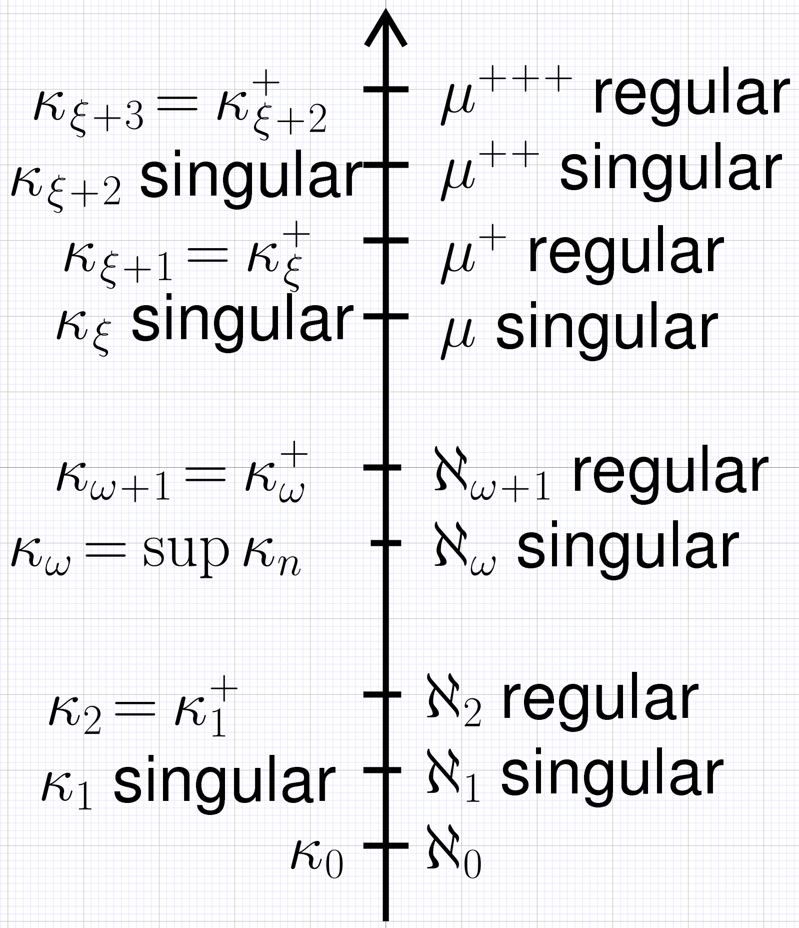
Proof. $\subseteq \checkmark$.
$\supseteq$. Let $\xi \notin X$. Let $q \in G$: $q \Vdash \check{\xi} \notin t(\check{\vec{z}}\,, \dot{g}_{\nu_0}, \ldots, \dot{g}_{\nu_{n-1}}, \dot{S})$. Set $r = q \restriction \{\nu_0, \ldots, \nu_{n-1}\}$.
Consider $p \in P, p \restriction \{\nu_0, \ldots, \nu_{n-1}\} \supseteq r$. By weak homogeneity, $\neg p \Vdash \check{\xi} \notin t(\check{\vec{z}}\,, \dot{g}_{\nu_0}, \ldots, \dot{g}_{\nu_{n-1}}, \dot{S})$.
Hence $\xi \notin$ RHS.

qed.

Cardinal preservation.

The extensions $V[G_{\nu_0} \times \ldots \times G_{\nu_{n-1}}]$ do not collapse the κ_i . Hence N does not collapse the κ_i .


$\kappa_{\xi+3} = \kappa_{\xi+2}^+$
μ^{+++} regular
$\kappa_{\xi+2}$ singular
μ^{++} singular
$\kappa_{\xi+1} = \kappa_\xi^+$
μ^+ regular
κ_ξ singular
μ singular
$\kappa_{\omega+1} = \kappa_\omega^+$
$\aleph_{\omega+1}$ regular
$\kappa_\omega = \sup \kappa_n$
$\aleph_\omega$ singular
$\kappa_2 = \kappa_1^+$
$\aleph_2$ regular
κ_1 singular
$\aleph_1$ singular
κ_0
$\aleph_0$


$\kappa_{\xi+3} = \kappa_{\xi+2}^{+}$
μ^{+++} regular
$\kappa_{\xi+2}$ singular
μ^{++} singular
$\kappa_{\xi+1} = \kappa_{\xi}^{+}$
μ^{+} regular
κ_{ξ} singular
μ singular
$\kappa_{\omega+1} = \kappa_{\omega}^{+}$
$\aleph_{\omega+1}$ regular
$\kappa_{\omega} = \sup \kappa_n$
$\aleph_{\omega}$ singular
$\kappa_2 = \kappa_1^{+}$
$\aleph_2$ regular
κ_1 singular
$\aleph_1$ singular
κ_0
$\aleph_0$

Thank You!