Ordinal Oriented Set Theory

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Set Theory, Classical and Constructive

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Leopold Kronecker

Talk in Berlin, 1886:

God created the integers, everything else is made by man.

(*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*)

Georg Cantor

Mitteilungen zur Lehre vom Transfiniten, 1887 and 1888

One of the most important challenges of set theory is to determine the sets which exist in all of nature; I was lead there by the development of the notion of ***ordinal number***

Eine der wichtigsten Aufgaben der Mengenlehre
... besteht in der Forderung, die in der Gesamtnatur,
soweit sie sich unsrer Erkenntnis aufschließt,
vorkommenden Mannigfaltigkeiten zu bestimmen;
dazu bin ich durch die Ausbildung ... des
Ordnungszahlbegriffs gelangt.

Cantor: Ordinals are generated by generating principles

- adjunction of a further unit

- taking the limit of previously defined numbers

- some limiting principle prevents to take the limit of all numbers

$$0, 1, 2, ..., \infty, \infty + 1, \infty + 2, ..., \infty + \infty, ...$$

Ordinals are the fundamental tool of Zermelo-Fraenkel set theory

- induction on the class Ord of all ordinals

- recursion on Ord: define $F\colon \mathrm{Ord} \to V$ by $\forall \alpha\, F(\alpha) = G(F \upharpoonright \alpha)$

- $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for limit λ

- $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$ is the (Zermelo-Fraenkel) universe of sets

Contents

- sets of ordinals in models of set theory

- axiomatising the class of sets of ordinals: $\mathbb{SO}$

- constructibility theory

- constructibility theory in $\mathbb{SO}$, using a recursive truth predicate on the ordinals

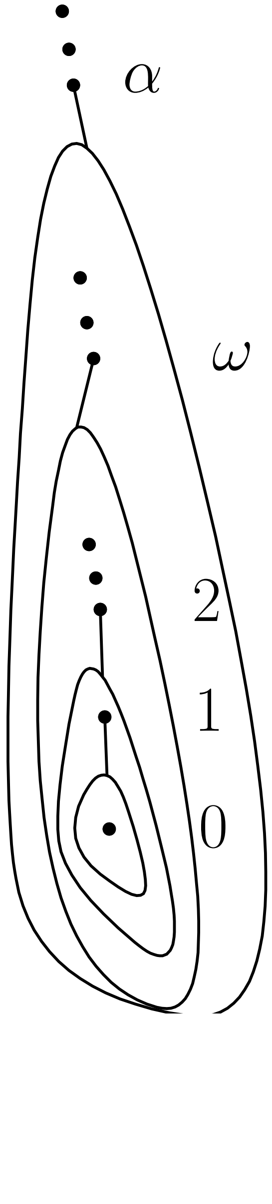
- computing the truth predicate by a “Turing” machine

- constructibility $\sim$ ordinal computability

- discussion: constructive set theory / computable set theory

von Neumann ordinals

- $0=\emptyset$, $1=\{0\}$, $2=\{0,1\}$,..., $\omega=\{0,1,2,\ldots\}$,...,$\alpha=\{\beta|\beta<\alpha\}$



α
ω
2
1
0

Ordinal arithmetic

- extend the recursive definitions of $+$ and $\times$ to all ordinals

- $\alpha+0=\alpha$, $\alpha+(\beta+1)=(\alpha+\beta)+1$, $\alpha+\lambda=\bigcup_{\beta<\lambda}(\alpha+\beta)$ for limit ordinals λ

- $\alpha\times 0=0$, $\alpha\times(\beta+1)=(\alpha\times\beta)+\alpha$, $\alpha\times\lambda=\bigcup_{\beta<\lambda}(\alpha\times\beta)$ for limit ordinals λ

- ordinal arithmetic allows pairing of ordinals: the function

$$(\alpha,\beta)\mapsto(\alpha+\beta)\times(\alpha+\beta)+\alpha$$

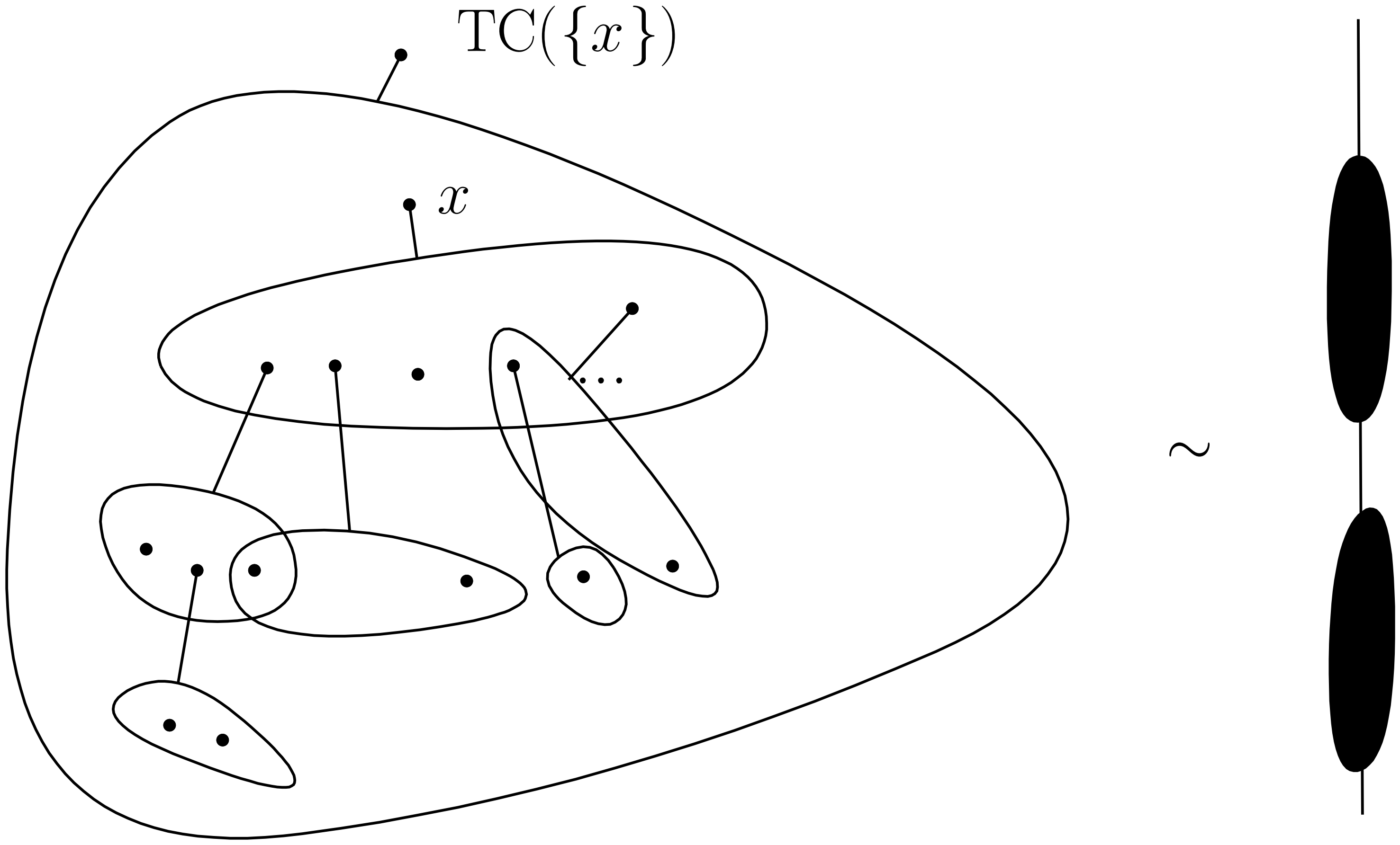
is injective

Sets of ordinals

- Every model M of ZFC set theory is determined by its class $M \cap \mathcal{P}(\mathrm{Ord})$ of sets of ordinals (folklore)

- by pairing, (set-sized) relations on ordinals can be coded as sets of ordinals

- idea: a set x is coded by a relation on ordinals isomorphic to $(\mathrm{TransitiveClosure}(\{x\}), \in)$



TC({x})
x
...
~

The theory SO

1. *Well-ordering axiom (WO)*:
$\forall\alpha,\beta,\gamma(\neg\alpha<\alpha\wedge(\alpha<\beta\wedge\beta<\gamma\rightarrow\alpha<\gamma)\wedge$
$(\alpha<\beta\vee\alpha=\beta\vee\beta<\alpha))\wedge$
$\forall a(\exists\alpha(\alpha\in a)\rightarrow\exists\alpha(\alpha\in a\wedge\forall\beta(\beta<\alpha\rightarrow\neg\beta\in a)))$;

2. *Axiom of infinity (INF)* (existence of a limit ordinal):
$\exists\alpha(\exists\beta(\beta<\alpha)\wedge\forall\beta(\beta<\alpha\rightarrow\exists\gamma(\beta<\gamma\wedge\gamma<\alpha)))$;

3. *Axiom of extensionality (EXT)*: $\forall a,b(\forall\alpha(\alpha\in a\leftrightarrow\alpha\in b)\rightarrow a=b)$;

4. *Initial segment axiom (INI)*: $\forall\alpha\exists a\forall\beta(\beta<\alpha\leftrightarrow\beta\in a)$;

5. *Boundedness axiom (BOU)*: $\forall a \exists \alpha \forall \beta (\beta \in a \rightarrow \beta < \alpha)$;

The theory SO

1. *Pairing axiom (GPF)* (Gödel Pairing Function):
$\forall \alpha, \beta, \gamma (g(\beta, \gamma) \leq \alpha \leftrightarrow \forall \delta, \epsilon ((\delta, \epsilon) <^* (\beta, \gamma) \rightarrow g(\delta, \epsilon) < \alpha))$.
Here $(\alpha, \beta) <^* (\gamma, \delta)$ stands for
$\exists \eta, \theta (\eta = \max(\alpha, \beta) \wedge \theta = \max(\gamma, \delta) \wedge (\eta < \theta \vee (\eta = \theta \wedge \alpha < \gamma) \vee (\eta = \theta \wedge \alpha = \gamma \wedge \beta < \delta)))$,
where $\gamma = \max(\alpha, \beta)$ abbreviates $(\alpha > \beta \wedge \gamma = \alpha) \vee (\alpha \leq \beta \wedge \gamma = \beta)$;

2. *Surjectivity of pairing (SUR)*: $\forall \alpha \exists \beta, \gamma (\alpha = g(\beta, \gamma))$;

3. *Axiom schema of separation (SEP)*: For all L_{SO}-for-

mulae $\phi(\alpha, P_1, ..., P_n)$, where $P_1, ..., P_n$ are variables for ordinals or sets of ordinals, postulate:
$\forall P_1, ..., P_n \forall a \exists b \forall \alpha(\alpha \in b \leftrightarrow \alpha \in a \wedge \phi(\alpha, P_1, ..., P_n))$;

4. *Axiom schema of replacement (REP)*: For all L_{SO}-formulae $\phi(\alpha, \beta, P_1, ..., P_n)$, where $P_1, ..., P_n$ are variables for ordinals or sets of ordinals, postulate:
$\forall P_1, ..., P_n(\forall \xi, \zeta_1, \zeta_2(\phi(\xi, \zeta_1, P_1, ..., P_n) \wedge \phi(\xi, \zeta_2, P_1, ..., P_n) \rightarrow \zeta_1 = \zeta_2) \rightarrow$
$\forall a \exists b \forall \zeta(\zeta \in b \leftrightarrow \exists \xi \in a \phi(\xi, \zeta, P_1, ..., P_n)))$;

5. *Powerset axiom (POW)*:
$\forall a \exists b(\forall z(\exists \alpha(\alpha \in z) \wedge \forall \alpha(\alpha \in z \rightarrow \alpha \in a) \rightarrow \exists \xi \forall \beta(\beta \in z \leftrightarrow g(\beta, \xi) \in b)))$.

Graphical proofs?

63=65?

Theorem. In a right-angled triangle (ABC) the square on the hypotenuse (AB) is equal to the sum of the squares on the other two sides (AC, BC).

Dem.---On the sides $A\,B$, $B\,C$, $C\,A$ describe squares [xlvi.]. Draw $C\,L$ parallel to $A\,G$. Join $C\,G$, $B\,K$. Then because the angle $A\;C\;B$ is right (hyp.), and $A\;C\;H$ is right, being the angle of a square, the sum of the angles $A\,C\,B$, $A\,C\,H$ is two right angles; therefore $B\,C$, $C\,H$ are in the same right line [xiv.]. In like manner $A\,C$, $C\,D$ are in the same right line. Again, because BAG is the angle of a square it is a right angle: in like manner $C\;A\;K$ is a right angle. Hence $B\;A\;G$ is equal to $C\;A\;K$: to each add $B\;A\;C$, and we get the angle $C\,A\,G$ equal to $K\,A\,B$. Again, since $B\,G$ and $C\,K$ are squares, $B\,A$ is equal to $A\,G$, and $C\,A$ to $A\,K$. Hence the two triangles $C\,A\,G$, $K\,A\,B$ have the sides $C\,A$, $A\,G$ in one respectively equal to the sides $K\,A$, $A\,B$ in the other, and the contained angles $C\,A\,G$, $K\,A\,B$ also equal. Therefore [iv.] the triangles are equal; but the parallelogram $A\,L$ is double of the triangle $C\,A\,G$ [xli.], because they are on the same base $A\,G$, and between the same parallels $A\;G$ and $C\;L$. In like manner the parallelogram $A\;H$ is double of the triangle $K\;A\;B$, because they are on the same base $A\,K$, and between the same parallels $A\,K$ and $B\,H$; and since doubles of equal things are equal (Axiom vi.), the parallelogram $A\,L$ is equal to $A\,H$. In like manner it can be proved that the parallelogram $B\;L$ is equal to $B\;D$. *Hence the whole square $A\;F$ is equal to the sum of the two squares $A\,H$ and $B\,D$.*

Dem.---On the sides $A B$, $B C$, $C A$ describe squares [xlvi.]. Draw $C L$ parallel to $A G$. Join $C G$, $B K$. Then because the angle $A C B$ is right (hyp.), and $A C H$ is right, being the angle of a square, the sum of the angles $A C B$, $A C H$ is two right angles; therefore $B C$, $C H$ are in the same right line [xiv.]. In like manner $A C$, $C D$ are in the same right line. Again, because BAG is the angle of a square it is a right angle: in like manner CAK is a right angle. Hence $B A G$ is equal to $C A K$: to each add $B A C$, and we get the angle $C A G$ equal to $K A B$. Again, since $B G$ and $C K$ are squares, $B A$ is equal to $A G$, and $C A$ to $A K$. Hence the two triangles $C A G$, $K A B$ have the sides $C A$, $A G$ in one respectively equal to the sides $K A$, $A B$ in the other, and the contained angles $C A G$, $K A B$ also equal. Therefore [iv.] the triangles are equal; but the parallelogram $A L$ is double of the triangle $C A G$ [xli.], because they are on the same base $A G$, and between the same parallels $A G$ and $C L$. In like manner the parallelogram $A H$ is double of the triangle $K A B$, because they are on the same base $A K$, and between the same parallels $A K$ and $B H$; and since doubles of equal things are equal (Axiom vi.), the parallelogram $A L$ is equal to $A H$. In like manner it can be proved that the parallelogram $B L$ is equal to $B D$. *Hence the whole square $A F$ is equal to the sum of the two squares $A H$ and $B D$.*

Dem.---On the sides AB, BC, CA describe squares [xlvi.]. Draw CL parallel to AG. Join CG, BK. Then because the angle ACB is right (hyp.), and ACH is right, being the angle of a square, the sum of the angles ACB, ACH is two right angles; therefore BC, CH are in the same right line [xiv.]. In like manner AC, CD are in the same right line. Again, because BAG is the angle of a square it is a right angle: in like manner CAK is a right angle. Hence BAG is equal to CAK: to each add BAC, and we get the angle CAG equal to KAB. Again, since BG and CK are squares, BA is equal to AG, and CA to AK. Hence the two triangles CAG, KAB have the sides CA, AG in one respectively equal to the sides KA, AB in the other, and the contained angles CAG, KAB also equal. Therefore [iv.] the triangles are equal; but the parallelogram AL is double of the triangle CAG [xli.], because they are on the same base AG, and between the same parallels AG and CL. In like manner the parallelogram AH is double of the triangle KAB, because they are on the same base AK, and between the same parallels AK and BH; and since doubles of equal things are equal (Axiom vi.), the parallelogram AL is equal to AH. In like manner it can be proved that the parallelogram BL is equal to BD. *Hence the whole square AF is equal to the sum of the two squares AH and BD.*

Dem.---On the sides $A\,B$, $B\,C$, $C\,A$ describe squares [xlvi.]. Draw $C\,L$ parallel to $A\,G$. Join $C\,G$, $B\,K$. Then because the angle $A\;C\;B$ is right (hyp.), and $A\;C\;H$ is right, being the angle of a square, the sum of the angles $A\,C\,B$, $A\,C\,H$ is two right angles; therefore $B\,C$, $C\,H$ are in the same right line [xiv.]. In like manner $A\,C$, $C\,D$ are in the same right line. Again, because $B\,A\,G$ is the angle of a square it is a right angle: in like manner $C\;A\;K$ is a right angle. Hence $B\;A\;G$ is equal to $C\;A\;K$: to each add $B\;A\;C$, and we get the angle $C\,A\,G$ equal to $K\;A\;B$. Again, since $B\,G$ and $C\,K$ are squares, $B\,A$ is equal to $A\,G$, and $C\,A$ to $A\,K$. Hence the two triangles $C\;A\;G$, $K\;A\;B$ have the sides $C\,A$, $A\,G$ in one respectively equal to the sides $K\,A$, $A\,B$ in the other, and the contained angles $C\,A\,G$, $K\,A\,B$ also equal. Therefore [iv.] the triangles are equal; but the parallelogram $A\;L$ is double of the triangle $C\;A\;G$ [xli.], because they are on the same base $A\;G$, and between the same parallels $A\;G$ and $C\;L$. In like manner the parallelogram $A\;H$ is double of the triangle $K\;A\;B$, because they are on the same base $A\;K$, and between the same parallels $A\;K$ and $B\,H$; and since doubles of equal things are equal (Axiom vi.), the parallelogram $A\,L$ is equal to $A\,H$. In like manner it can be proved that the parallelogram $B\;L$ is equal to $B\;D$. *Hence the whole square $A\;F$ is equal to the sum of the two squares $A\,H$ and $B\,D$.*

1.	Φ_{Gr}	$\neg \circ v_0\, e \equiv v_0$		$\neg \exists v_0 \neg \circ v_0\, e \equiv v_0$	VR
2.	Φ_{Gr}	$\neg \circ v_0\, e \equiv v_0$		$\neg \circ v_0\, e \equiv v_0$	VR
3.	Φ_{Gr}	$\neg \circ v_0\, e \equiv v_0$		$\exists v_0 \neg \circ v_0\, e \equiv v_0$	∃S auf 2
4.	Φ_{Gr}			$\circ v_0\, e \equiv v_0$	WR auf 1,3
5.				$(v_2 \equiv \circ v_0\, e)\frac{\circ v_0\, e}{v_2}$	$(\equiv)$
6.		$\circ v_0\, e \equiv v_0$		$(v_2 \equiv \circ v_0\, e)\frac{v_0}{v_2}$	Sub auf 5
7.	Φ_{Gr}	$\circ v_0\, e \equiv v_0$		$v_0 \equiv \circ v_0\, e$	AR auf 6
8.	Φ_{Gr}			$v_0 \equiv \circ v_0\, e$	KS auf 4,7
9.	Φ_{Gr}	$v_0 \equiv e$		$v_0 \equiv e$	VR
10.	Φ_{Gr}	$v_0 \equiv e$		$(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	∨S auf 9
11.	Φ_{Gr}	$\neg v_0 \equiv e$		$(\neg v_2 \equiv e)\frac{v_0}{v_2}$	VR
12.	Φ_{Gr}	$\neg v_0 \equiv e$	$v_0 \equiv \circ v_0\, e$	$(\neg v_2 \equiv e)\frac{\circ v_0\, e}{v_2}$	Sub auf 11
13.	Φ_{Gr}	$\neg v_0 \equiv e$	$v_0 \equiv \circ v_0\, e$	$\neg \circ v_0\, e \equiv e$	12
14.	Φ_{Gr}	$\neg v_0 \equiv e$		$v_0 \equiv \circ v_0\, e$	AR auf 8
15.	Φ_{Gr}	$\neg v_0 \equiv e$		$\neg \circ v_0\, e \equiv e$	KS auf 14
16.	Φ_{Gr}	$\neg v_0 \equiv e$		$(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	∨S auf 15
17.	Φ_{Gr}			$(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	FU auf 10,16
18.	Φ_{Gr}	$\neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	$\neg\neg\exists v_0 \neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	$(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	AR auf 17
19.	Φ_{Gr}	$\neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	$\neg\neg\exists v_0 \neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	$\neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	VR
20.	Φ_{Gr}	$\neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$		$\neg\exists v_0 \neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	WR auf 18,19
21.	Φ_{Gr}	$\exists v_0 \neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$		$\neg\exists v_0 \neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	∃A auf 20
22.	Φ_{Gr}	$\neg\exists v_0 \neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$		$\neg\exists v_0 \neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	VR
23.	Φ_{Gr}			$\neg\exists v_0 \neg(\neg \circ v_0\, e \equiv e \vee v_0 \equiv e)$	FU auf 21,22

1.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$	$\neg\exists v_0 \neg \circ v_0 e \equiv v_0$	VR
2.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$	$\neg \circ v_0 e \equiv v_0$	VR
3.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$	$\exists v_0 \neg \circ v_0 e \equiv v_0$	$\exists$S auf 2
4.	Φ_{Gr}		$\circ v_0 e \equiv v_0$	WR auf 1,3

1.	Φ_{Gr}	$\neg \circ v_0\, e \equiv v_0$	$\neg\exists v_0 \neg \circ v_0\, e \equiv v_0$	VR
2.	Φ_{Gr}	$\neg \circ v_0\, e \equiv v_0$	$\neg \circ v_0\, e \equiv v_0$	VR
3.	Φ_{Gr}	$\neg \circ v_0\, e \equiv v_0$	$\exists v_0 \neg \circ v_0\, e \equiv v_0$	∃S auf 2
4.	Φ_{Gr}		$\circ v_0\, e \equiv v_0$	WR auf 1,3

$\vdots$

1.	Φ_{Gr}	$\neg\circ v_0\, e \equiv v_0$	$\neg\exists v_0 \neg\circ v_0\, e \equiv v_0$	VR
2.	Φ_{Gr}	$\neg\circ v_0\, e \equiv v_0$	$\neg\circ v_0\, e \equiv v_0$	VR
3.	Φ_{Gr}	$\neg\circ v_0\, e \equiv v_0$	$\exists v_0 \neg\circ v_0\, e \equiv v_0$	∃S auf 2
4.	Φ_{Gr}		$\circ v_0\, e \equiv v_0$	WR auf 1,3

$\vdots$

1.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$	$\neg \exists v_0 \neg \circ v_0 e \equiv v_0$	VR
2.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$	$\neg \circ v_0 e \equiv v_0$	VR
3.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$	$\exists v_0 \neg \circ v_0 e \equiv v_0$	∃S auf 2
4.	Φ_{Gr}		$\circ v_0 e \equiv v_0$	WR auf 1,3

$\vdots$

$\vdots$

21.	Φ_{Gr}	$\exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$\neg\exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	∃A auf 20
22.	Φ_{Gr}	$\neg\exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$\neg\exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	VR
23.	Φ_{Gr}		$\neg\exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	FU auf 21,22

Formal proofs - derivations

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \psi \quad \varphi}$$

$$\frac{}{\Gamma \quad \varphi \quad \varphi}$$

$$\frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \varphi \to \psi}$$

$$\frac{\begin{array}{l}\Gamma \quad \varphi \\ \Gamma \quad \varphi \to \psi\end{array}}{\Gamma \quad \psi}$$

$$\frac{\begin{array}{l}\Gamma \quad \varphi \\ \Gamma \quad \neg\varphi\end{array}}{\Gamma \quad \bot}$$

$$\frac{\begin{array}{l}\Gamma \quad \varphi \\ \Gamma \quad \neg\varphi\end{array}}{\Gamma \quad \bot}$$

$$\frac{\Gamma \quad \neg\varphi \quad \bot}{\Gamma \quad \varphi}$$

$$\frac{\Gamma \quad \varphi\frac{y}{x}}{\Gamma \quad \forall x\varphi},$$

if $y \notin \mathrm{free}(\Gamma \cup \{\forall x\varphi\})$

$$\frac{\Gamma \quad \forall x\varphi}{\Gamma \quad \varphi\frac{t}{x}}$$

$$\frac{}{\Gamma \quad t \equiv t}$$

$$\frac{\begin{array}{l}\Gamma \quad \varphi\frac{t}{x} \\ \Gamma \quad t \equiv t'\end{array}}{\Gamma \quad \varphi\frac{t'}{x}}$$

Formal proofs - derivations

- derivations are formed by repeated applications of (simple) syntactic rules

- whether a formal text is a derivation can (easily) be checked algorithmically

Formal proofs - derivations

N. Bourbaki:

If formalized mathematics were as simple as the game of chess, then once our chosen formalized language had been described there would remain only the task of writing out our proofs in this language, [...] But the matter is far from being as simple as that, and no great experience is necessary to perceive that such a project is absolutely unrealizable: the tiniest proof at the beginnings of the Theory of Sets would already require several hundreds of signs for its complete formalization. [...] formalized mathematics cannot in practice be written down in full, [...] We shall therefore very quickly abandon formalized mathematics, [...]

Formal proofs

Saunders Mac Lane:

As to precision, we have now stated an absolute standard of rigor: A mathematical proof is rigorous when it is (or could be) written out in the first-order predicate language $L(\in)$ as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules. [...] When a proof is in doubt, its repair is usually a partial approximation to the fully formal version.

Computer-supported formal proofs

J. McCarthy:

Checking mathematical proofs is potentially one of the most interesting and useful applications of automatic computers. ... Proofs to be checked by computer may be briefer and easier to write than the informal proofs acceptable to mathematicians. This is because the computer can be asked to do much more work to check each step than a human is willing to do, and this permits longer and fewer steps.

McCarthy, J. "Computer Programs for Checking Mathematical Proofs," Proceedings of the Symposium in Pure Math, Recursive Function Theory,Volume V, pages 219-228, AMS, Providence, RI, 1962.

Automatic proof checker

Automath (~1967)

N.G. de Bruijn

From the Automath formalization of E. Landau, *Grundlagen der Analysis*, 1930
by L. S. van Benthem Jutting, 1979:

```
ic:=pli(0,1rl):complex
+10300
t1:=tsis12a(0,1rl,0,1rl):is(ts(ic,ic),pli(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),pl"r"(ts"r"(0,1rl),
ts"r"(1rl,0))))
t2:=tris(real,mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),m0"r"(ts"r"(1rl,1rl)),m0"r"(1rl),pl01(ts"r"(0,0),
m0"r"(ts"r"(1rl,1rl)),ts01(0,0,refis(real,0))),ism0"r"(ts"r"(1rl,1rl),1rl,satz195(1rl))):
is"r"(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),m0"r"(1rl))
t3:=tris(real,pl"r"(ts"r"(0,1rl),ts"r"(1rl,0)),ts"r"(1rl,0),0,pl01(ts"r"(0,1rl),ts"r"(1rl,0),
ts01(0,1rl,refis(real,0))),ts02(1rl,0,refis(real,0))):is"r"(pl"r"(ts"r"(0,1rl),ts"r"(1rl,0)),0)
t4:=isrecx12(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),m0"r"(1rl),pl"r"(ts"r"(0,1rl),
ts"r"(1rl,0)),0,t2,t3):is(pli(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),
pl"r"(ts"r"(0,1rl),ts"r"(1rl,0))),cofrl(m0"r"(1rl)))
t5:=satz298j(1rl):is(cofrl(m0"r"(1rl)),m0(1c))
-10300
satz2300:=tr3is(cx,ts(ic,ic),pli(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),
pl"r"(ts"r"(0,1rl),ts"r"(1rl,0))),cofrl(m0"r"(1rl)),m0(1c),t1".10300",t4".10300",t5".10300"):
is(ts(ic,ic),m0(1c))
```

The MIZAR system (1973 -) of Andrzej Trybulec

Language modeled after
“mathematical vernacular”

Natural deduction style

Automatic proof checker

Large mathematical library

Journal
Formalized Mathematics

www.mizar.org

MIZAR example: Proof of Pythagoras

```
theorem for p1,p2,p3 st p1<>p2 & p3<>p2 &
    (angle(p1,p2,p3)=PI/2 or angle(p1,p2,p3)=3/2*PI) holds
    (|.p1-p2.|^2+|.p3-p2.|^2=|.p1-p3.|^2
    proof let p1,p2,p3; assume A1: p1<>p2 & p3<>p2 &
    (angle(p1,p2,p3)=PI/2 or angle(p1,p2,p3)=3/2*PI);
  then A2: euc2cpx(p1)<> euc2cpx(p2) by Th6;
  A3: euc2cpx(p3)<> euc2cpx(p2) by A1,Th6;
  A4: euc2cpx(p1)-euc2cpx(p2)=euc2cpx(p1-p2) by Th19;
  A5: euc2cpx(p3)-euc2cpx(p2)=euc2cpx(p3-p2) by Th19;
  A6: euc2cpx(p1)-euc2cpx(p3)=euc2cpx(p1-p3) by Th19;
  A7: angle(p1,p2,p3)=angle(euc2cpx(p1),euc2cpx(p2),euc2cpx(p3))
by Def4;
  A8: |.euc2cpx(p1-p2).|=|.p1-p2.| by Th31;
  A9: |.euc2cpx(p3-p2).|=|.p3-p2.| by Th31;
   |.euc2cpx(p1-p3).|=|.p1-p3.| by Th31;
  hence thesis by A1,A2,A3,A4,A5,A6,A7,A8,A9,COMPLEX2:91;
end;
```

Derivations

- (Euclid)

- (Hilbert-style) calculi

- Automath

- MIZAR

What is a mathematical proof?

- description of the/some mathematical “reality”?
- argumentative text about the/some mathematical “reality”?
- argumentative text within some system of initial assumptions (axioms)?
- Wittgenstein: ...... ?
- abbreviation for some (long) formal derivation?
- recipe for building a formal derivation if required?
- a formal derivation in some very rich formal system (Montague: English as a formal language)?

Jody Azzouni: The derivation-indicator view of mathematical practice

ABSTRACT. A version of Formalism is vindicated: Ordinary mathematical proofs indicate (one or another) mechanically checkable derivation of theorems from the assumptions those ordinary mathematical proofs presuppose. The indicator view explains why mathematicians agree so readily on results established by proofs in ordinary language that are (palpably) not mechanically checkable. Mechanically checkable derivations in this way structure ordinary mathematical practice without its being the case that ordinary mathematical proofs can be ’reduced to’ such derivations. In this way, one threat to formalist-style positions is removed: Platonic objects aren’t needed to explain how mathematicians understand the import of ordinary mathematical proofs. (Philosophia Mathematica, 2004)

Derivation-indication

N. Bourbaki:

If formalized mathematics were as simple as the game of chess, ...

... there would remain only the task of writing out our proofs in this language, ...

Derivation-indication

Saunders Mac Lane:

As to precision, we have now stated an absolute standard of rigor: A mathematical proof is rigorous when it is (or could be) written out in the first-order predicate language $L(\in)$ as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules. [...] When a proof is in doubt, its repair is usually a partial approximation to the fully formal version.

Derivation-indication

- Mathematicians agree that proofs can be written out in increasingly formal detail

- This leads to a fully formal derivation after some (long) finite time

- The indicator function lies mainly in the natural language parts of proofs

- Can one identify indicators by natural language processing?

- Derivations may be derivations performed by an Automatic Theorem Prover (ATP).

The Naproche project: Natural language proof checking

- studies the syntax and semantics of the language of proofs, emphasizing natural language and natural argumentation aspects

- models natural language proofs using computer-supported methods of formal linguistics and formal logic

- “reverse engineering” approach to derivation-indication

- joint work with Bernhard Schröder, linguistics; Bonn, Essen, Cologne; www.naproche.net

- development of a mathematical authoring system with a LaTeX-quality graphical interface

The Naproche project: Natural language proof checking

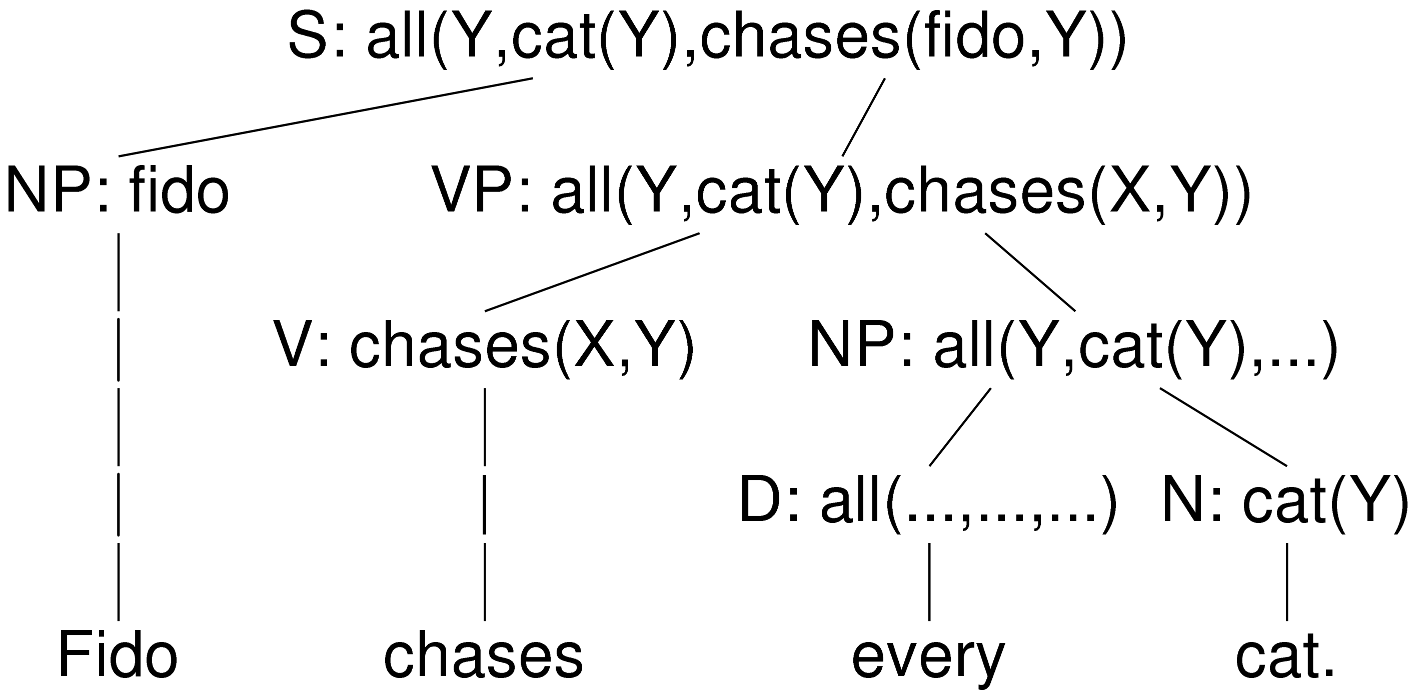
– To devise a strictly formal system for mathematics, implemented by computer, whose input language is an extensive part of the common mathematical language, and whose proof style is close to proof styles found in the mathematical literature.

Mathematical statements

"1 divides every integer." $\longleftrightarrow$ "Fido chases every cat."

Linguistic analysis

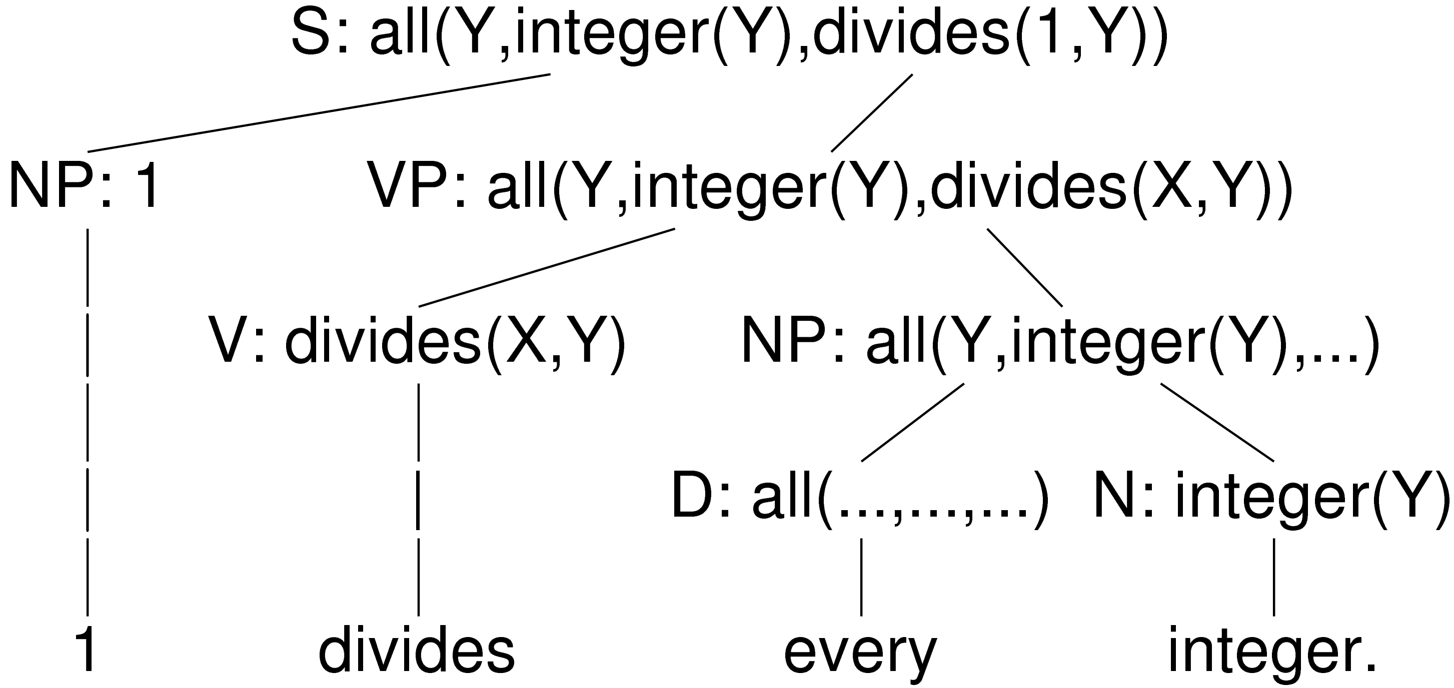
“Fido chases every cat.”



$\forall Y(\mathrm{cat}(Y) \rightarrow \mathrm{chases}(\mathrm{fido}, Y))$.

Linguistic analysis

“1 divides every integer.”



$\forall Y(\text{integer}(Y) \rightarrow 1|Y)$.

Layers of the Naproche system:

$\downarrow$ Standard or web editor

TeX-style input text

$\updownarrow$ Natural language processing (NLP)

Proof representation structure (PRS)

$\updownarrow$ First-order translation

First-order logic format (TPTP)

$\updownarrow$ Proof checker or automatic theorem prover (ATP)

“Accepted”/“Not accepted”, with error messages

E. Landau, *Grundlagen der Analysis*, 1930: Theorem 30

Components of the Naproche system: linguistic analysis

– standard analysis by a Prolog Definite Clause Grammar (DCG), the grammar defines a controlled natural language for mathematics (CNL), i.e. a formal subset of the common mathematical language

– translation into a formal semantics (without ambiguity)

Components of the Naproche system: linguistic analysis

- formal semantics: proof representation structures (PRS), extending discourse representation structures (DRS)

- DRS: tool for anaphor resolution (Let $\underline{x}$ be a set. It is ...) and for interpretation of natural language quantification (Every prime number is positive; a prime number is positive)

- PRS, moreover, represent global text structurings: Theorem / Proof, introductions and retractions of assumptions

Components of the Naproche system: Checking logical correctness

- translating the PRS conditions into some first-order format
- use TPTP-format (Thousands of Problems for Theorem Provers)
- generate relevant premises for every condition
- automatic theorem prover (ATP) used to prove every condition from its relevant premises
- proof is accepted if ATP can prove every condition
- feedback of success/error messages

Results

– The Naproche system allows natural reformulation of (simple) mathematical texts

– some example texts and parts of Landau, Foundations of Analysis have been reformulated and checked

Chapter 1 from Landau in Naproche

by Merlin Carl, Marcos Cramer, Daniel Khlwein

May 4, 2010

Abstract

This is a reformulation of the first chapter of Landau's *Grundlagen der Analysis* in the Controlled Natural Language of Naproche. Talk about sets is still avoided. One consequence of this is that Axiom 5 (the induction axiom) cannot be formulated; instead we use an induction proof method.

Axiom 3: For every x, $x' \neq 1$.

Axiom 4: If $x' = y'$, then $x = y$.

Theorem 1: If $x \neq y$ then $x' \neq y'$.
Proof:
Assume that $x \neq y$ and $x' = y'$. Then by axiom 4, $x = y$. Qed.

Theorem 2: For all x $x' \neq x$.
Proof:
By axiom 3, $1' \neq 1$. Suppose $x' \neq x$. Then by theorem 1, $(x')' \neq x'$. Thus by induction, for all x $x' \neq x$. Qed.

Theorem 3: If $x \neq 1$ then there is a u such that $x = u'$.
Proof:
If $1 \neq 1$ then there is a u such that $1 = u'$.
Assume $x' \neq 1$. If $u = x$ then $x' = u'$. So there is a u such that $x' = u'$.
Thus by induction, if $x \neq 1$ then there is a u such that $x = u'$. Qed.

Definition 1:
Define $+$ recursively:
$x + 1 = x'$.
$x + y' = (x + y)'$.

Theorem 5: For all x, y, z, $(x + y) + z = x + (y + z)$.
Proof:
Fix x, y.
$(x + y) + 1 = (x + y)' = x + y' = x + (y + 1)$.
Assume that $(x + y) + z = x + (y + z)$. Then $(x + y) + z' = ((x + y) + z)' = (x + (y + z))' = x + (y + z)' = x + (y + z')$. So $(x + y) + z' = x + (y + z')$.
Thus by induction, for all z, $(x + y) + z = x + (y + z)$. Qed.

Lemma 4a: For all y, $1 + y = y'$.
Proof:
By definition 1, $1 + 1 = 1'$.
Suppose $1 + y = y'$. Then by definition 1, $1 + y' = (1 + y)'$. So $1 + y' = (y')'$.
Thus by induction, for all y $1 + y = y'$. Qed.

Possible applications

- Natural language interfaces to formal mathematics
- Mathematical authoring and checking tools
- writing texts that are simultaneously acceptable by human readers and formal mathematics systems ("Logic for men and machines")
- Tutorial applications: teaching how to prove

General issues

- Naproche attempts to implement parts of the derivation-indication approach to proofs
- natural language components serve as indicators
- there are natural(ly looking) proofs that are fully formal with respect to the Naproche system
- this defines a “fortified formalism”, using linguistic methods and computer implementations, which allows to view some natural proofs as fully formal
- can a “fortified formalism” help to mediate between the “two streams” in the philosophy of mathematics (formalistic / naturalistic)?

Thank You!