Moduli spaces of twisted k-differentials

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Abstract

Given a smooth curve C and some points $p_i \in C$, one can ask if there exists a meromorphic k-differential η on C with zeroes and poles of specified orders exactly at the points p_i. This condition cuts out a closed subset of $\mathcal{M}_{g,n}$. In [1], Farkas and Pandharipande propose a natural way to extend this to a closed subset of $\overline{\mathcal{M}}_{g,n}$. One can use deformation theory to compute the dimension of its components. Moreover, there is a conjectural formula of its (weighted) fundamental class in terms of a tautological cycle studied by Pixton.

The space $\mathcal{H}_g^k(\mu)$

Let $g, n, k \geq 0$ and $\mu = (m_1, \ldots, m_n)$ an integer partition of $k(2g-2)$. Inside the moduli space $\mathcal{M}_{g,n}$ of smooth genus g curves C with n distinct marked points $p_1, \ldots, p_n \in C$, define

$$\mathcal{H}_g^k(\mu) = \left\{(C, p_1, \ldots, p_n) : \mathcal{O}\left(\sum_{i=1}^n m_i p_i\right) \cong \omega_C^{\otimes k}\right\}$$

This equality of line bundles is equivalent to the existence of a meromorphic k-differential η on C with $\mathrm{div}(\eta) = \Sigma_{i=1}^n m_i p_i$.

Example: $\mathcal{H}_2^2(2,2)$

We are looking for the set of genus 2 curves C with markings p, q such that $\mathcal{O}(2p + 2q) \cong \omega_C^{\otimes 2}$. But recall that every such C is hyperelliptic via its canonical map and for each $p \in C$ the hyperelliptic conjugate point $\bar{p}$ satisfies $\mathcal{O}(p + \bar{p}) \cong \omega_C$.
Then $\mathcal{H}_2^2(2,2)$ is the disjoint union of the sets

$$\mathcal{H}_2^1(1,1) = \{(C,p,q) : \bar{p} = q\}$$
$$\mathcal{H}_2^2(2,2)' := \{(C,p,q) : \bar{p} = p, \bar{q} = q\}$$

Note that the first set has codimension 1 in $\mathcal{M}_{2,2}$, while the second has codimension 2!
Exercise: Show that any $(C,p,q) \in \mathcal{H}_2^2(2,2)$ is in the union of the two sets above by constructing a meromorphic differential ξ with $\mathrm{div}(\xi) = 2p + q - \bar{q}$ and using the Residue theorem.

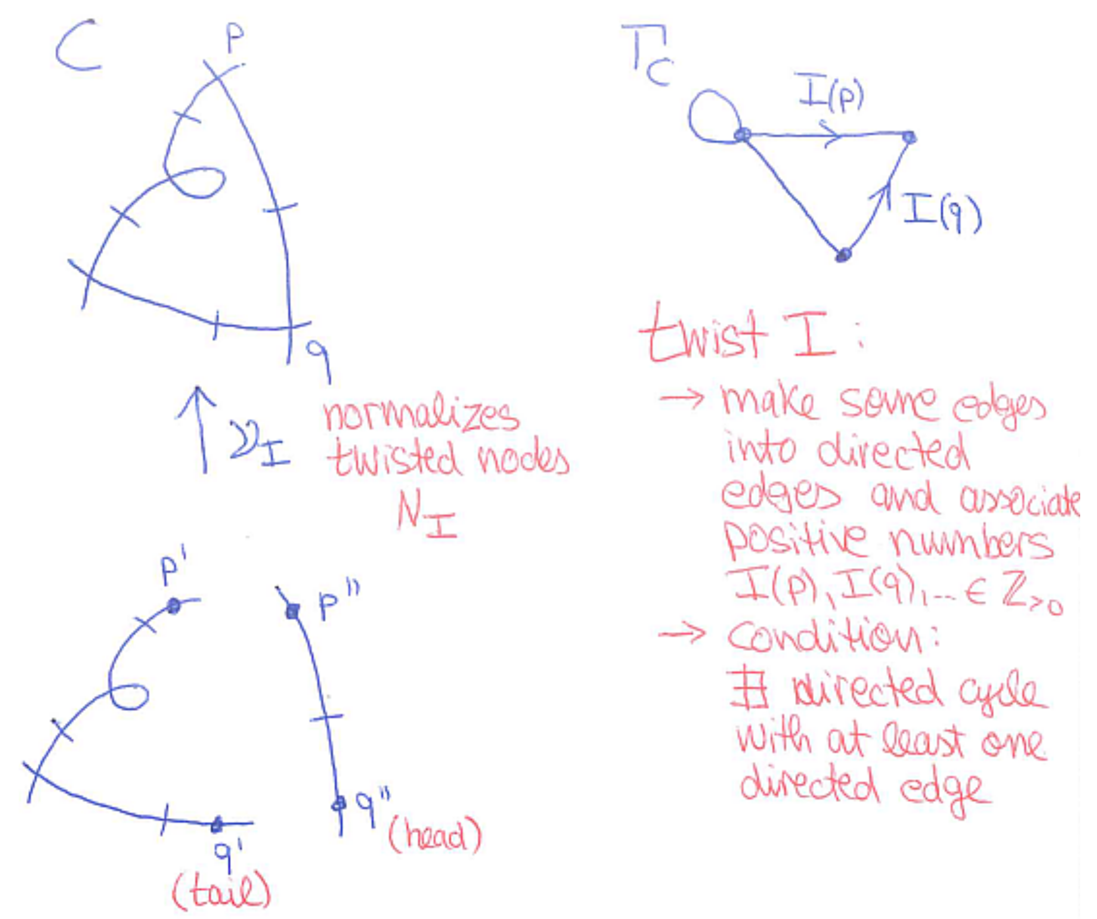
Compactifications of $\mathcal{H}_g^k(\mu)$

Taking the closure $\overline{\mathcal{H}}_g^k(\mu) \subset \overline{\mathcal{M}}_{g,n}$ is one way to compactify $\mathcal{H}_g^k(\mu)$, but it lacks a simple modular interpretation.
Instead, Farkas and Pandharipande proposed in [1] a different compactification - called the moduli space of twisted k-differentials. It is given by a condition $(*)$, which we describe below and on the right.

$$\widetilde{\mathcal{H}}_g^k(\mu) = \{(C, p_1, \ldots, p_n) : (*) \text{ is satisfied}\}$$

with
$(*)$: There exists a twist I on Γ_C with

$$\nu_I^* \mathcal{O}_C\left(\sum_{i=1}^n m_i p_i\right) \cong \nu_I^* \omega_C^{\otimes k} \otimes \mathcal{O}_{C_I}\left(\sum_{q \in N_I} I(q) \cdot (q' - q'')\right)$$



Theorem (Farkas-Pandharipande ($k = 1$,[1]), S. ($k > 1$,[2]))

For $k \geq 1$, the components of $\widetilde{\mathcal{H}}_g^k(\mu)$ have dimension $2g - 3 + n$ (codimension g), <u>except</u> if μ is divisible by k and nonnegative, in which case $\overline{\mathcal{H}}_g^1(\mu/k) \subset \widetilde{\mathcal{H}}_g^k(\mu)$ is a union of components of dimension $2g - 2 + n$ (codimension $g - 1$), with all other components of dimension $2g - 3 + n$. Moreover, the components of $\mathcal{H}_g^k(\mu) \subset \mathcal{M}_{g,n}$ are smooth of the dimensions given above.

Idea of proof

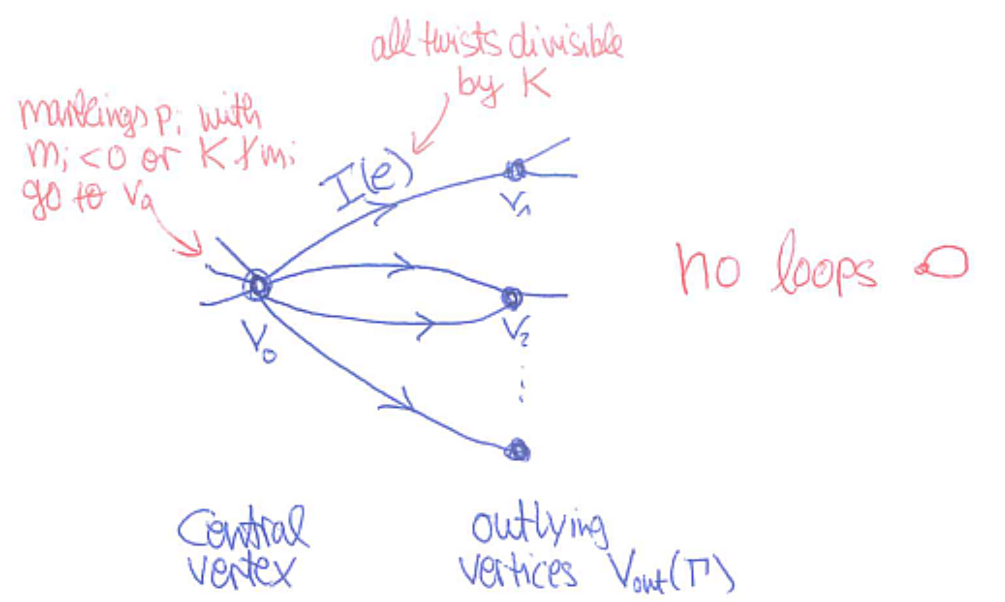
The components of $\widetilde{\mathcal{H}}_g^k(\mu)$ in the boundary of $\overline{\mathcal{M}}_{g,n}$ are parametrized by products of spaces $\overline{\mathcal{H}}_{g'}^{k'}(\mu')$ (see on the right) so by induction the main difficulty lies in analyzing the open part $\mathcal{H}_g^k(\mu)$.
For $\pi : \mathcal{J} \to \mathcal{M}_{g,n}$ the universal Jacobian of degree $k(2g-2)$, the set $\mathcal{H}_g^k(\mu)$ is the intersection of the sections $s_1 = \omega_C^{\otimes k}$ and $s_2 = \mathcal{O}(\Sigma_{i=1}^n m_i p_i)$. Thus its tangent space at a point $(C,p) = (C, p_1, \ldots, p_n) \in \mathcal{H}_g^k(\mu)$ is given by

$$T_{(C,p)}\mathcal{H}_g^k(\mu) = \ker T_{(C,p)}\mathcal{M}_{g,n} \xrightarrow{ds_1 - ds_2} T_{(C,p,\omega_C^{\otimes k})}\mathcal{J}$$

By deformation theory, both tangent spaces have natural expressions in terms of cohomology groups of sheaves on C and the map $ds_1 - ds_2$ is induced from a map of these sheaves. By analyzing kernel and cokernel of the sheaf map separately, one can compute the dimension of the kernel in cohomology.

Boundary components

Let Z be a component of $\widetilde{\mathcal{H}}_g^k(\mu)$ in the boundary, with generic dual graph Γ and generic twist I on Γ. From the known dimension of $\overline{\mathcal{H}}_g^k(\mu)$ and a general lower bound on the the dimension of $\widetilde{\mathcal{H}}_g^k(\mu)$, it follows that Γ, I must be of a particular form:



Denote by $C_{\Gamma,I}$ the union of components Z with given dual graph Γ and twist I.

Relation to Pixton's cycle

If $\widetilde{\mathcal{H}}_g^k(\mu)$ has pure codimension g, it makes sense to ask for its fundamental class. If one weights the various components $C_{\Gamma,I}$ by combinatorially defined integers, there is a conjectural formula for this weighted sum in terms of Pixton's cycle.

Conjecture A

For $k \geq 1$ and μ not of the form $\mu = k\mu'$ for a nonnegative partition μ' of $2g-2$, let $\tilde{\mu} = (m_1 + k, \ldots, m_n + k)$. Then we have

$$\sum_{\Gamma, I} \frac{\prod_{e \in E(\Gamma)} I(e)}{k^{|V_{\mathrm{out}}(\Gamma)|}} [C_{\Gamma,I}] = 2^{-g} P_g^{g,k}(\tilde{\mu}).$$

Here $P_g^{g,k}(\tilde{\mu})$ is an explicit sum of pushforwards under gluing maps

$$\xi_{\Gamma'} : \prod_{v \in V(\Gamma')} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}$$

of polynomials in κ and ψ-classes. It is defined purely combinatorially.
Using the fact that $[C_{\Gamma,I}]$ can itself be written as a gluing pushforward

$$[C_{\Gamma,I}] = \frac{1}{|\mathrm{Aut}(\Gamma)|}(\xi_\Gamma)_* \Big[\left[\overline{\mathcal{H}}_{g(v_0)}^k(\mu[v_0], -I[v_0] - k)\right] \cdot \prod_{v \in V_{\mathrm{out}}(\Gamma)} \left[\overline{\mathcal{H}}_{g(v)}^1\left(\frac{\mu[v]}{k}, \frac{I[v] - k}{k}\right)\right] \Big]$$

this can be used to determine all the classes $[\overline{\mathcal{H}}_g^k(\mu)]$.

References

[1] G. Farkas and R. Pandharipande.
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[2] J. Schmitt.
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