A compactification of the space of self-maps of $\mathbb{CP}^1$

Johannes Schmitt
ETH Zurich

Abstract

We will describe a parameter space for the morphisms $\mathbb{CP}^1 \to \mathbb{CP}^1$ and compactify it. Then we divide out the simultaneous action of $\mathrm{Aut}(\mathbb{CP}^1)$ on domain and target to obtain compactifications of the moduli space of self-maps of $\mathbb{CP}^1$. Several properties of this new space are given.

The space of morphisms $\mathbb{CP}^1 \to \mathbb{CP}^1$

We identify an algebraic map $\varphi : \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree d with its **graph**

$$\Gamma_\varphi = \{(x, \varphi(x)) : x \in \mathbb{CP}^1\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1.$$

Γ_φ is a complex curve of class $(1,d) \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z})$. Thus Γ_φ is the vanishing set of a global section s of the line bundle $\mathcal{O}(d,1)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$, unique up to scaling. This gives a unique element

$$[s] \in \mathbb{P}(H^0(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(d,1))) =: Z_d.$$

The set of $[s]$ obtained like this forms an open subset $\mathrm{Rat}_d \subset Z_d$.

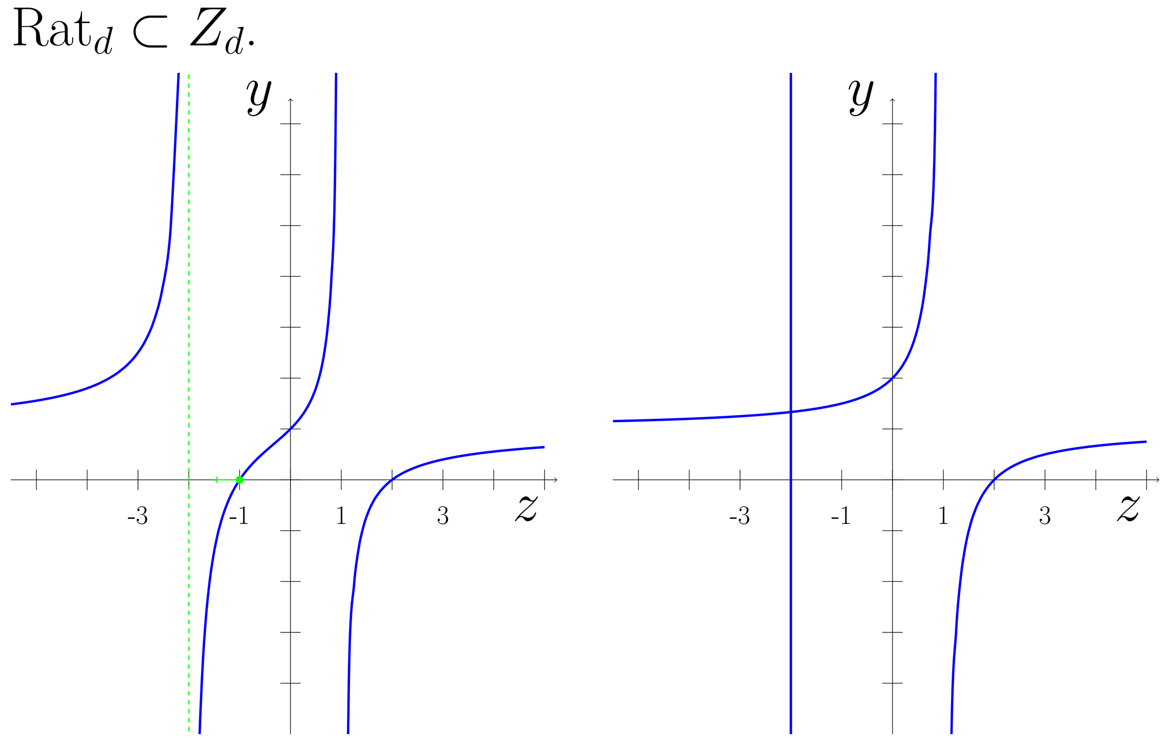

Figure 1: Degeneration in Z_2 as pole and zero collide

Alternatively, we can add to a graph Γ the structure of a **parametrization**. For this consider the moduli space

$$\overline{M}_{0,n}(\mathbb{CP}^1 \times \mathbb{CP}^1, (1,d)) =: Y_{d,n}$$

of stable maps $f : C \to \mathbb{CP}^1 \times \mathbb{CP}^1$ from an n-pointed genus 0 curve C. For such a map there is a unique $[s] \in Z_d$ with $f_*[C] = \mathrm{div}(s)$. This gives a natural morphism $j : Y_{d,n} \to Z_d$ and for $n = 0$ it induces an isomorphism from the locus of maps with $C = \mathbb{CP}^1$ to Rat_d.

The conjugation action of $\mathrm{PGL}_2(\mathbb{C})$

Identifying domain and target of a map $\varphi : \mathbb{CP}^1 \to \mathbb{CP}^1$ to be the *same* $\mathbb{CP}^1$ we want to study such maps up to *simultaneous* choice of coordinates on $\mathbb{CP}^1$. A coordinate change then corresponds to the **conjugation** by an element $\psi \in \mathrm{Aut}(\mathbb{CP}^1) = \mathrm{PGL}_2(\mathbb{C})$. The induced action

$$\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{Rat}_d \to \mathrm{Rat}_d, (\psi, \varphi) \mapsto \psi \circ \varphi \circ \psi^{-1}$$

extends naturally to Z_d and $Y_{d,n}$. On those (parametrized) graphs it is induced by the usual action of $\mathrm{PGL}_2(\mathbb{C})$ on the factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Stability conditions

We can then define the quotient of Z_d and $Y_{d,n}$ by this action using **Geometric Invariant Theory** (GIT). In order for the quotient to have an algebraic structure, we must restrict ourselves to points $Z_d^{ss} \subset Z_d$ and $Y_{d,n}^{ss} \subset Y_{d,n}$ that are **semistable**. This notion is defined canonically for Z_d and for d even it pulls back well to $Y_{d,n}$ via the map $j : Y_{d,n} \to Z_d$. Roughly a (parametrized) graph is (semi)stable iff none of the vertical components have multiplicity (as an algebraic cycle) greater than $d/2$. In particular $\mathrm{Rat}_d \subset Z_d^{ss}$.

Main definition

$$M_d := \mathrm{Rat}_d /\!\!/ \mathrm{PGL}_2(\mathbb{C}) \subset M_d^s := Z_d^{ss} /\!\!/ \mathrm{PGL}_2(\mathbb{C})$$
$$M(d,n) := Y_{d,n}^{ss} /\!\!/ \mathrm{PGL}_2(\mathbb{C})$$

First properties

Let $d \geq 2$ even and $n \geq 0$, then we know

- $M(d,n)$ is a **normal, projective variety** over $\mathbb{C}$
- the map $\phi : Y_{d,n}^{ss} \to M(d,n)$ is a **geometric quotient**
- $M(2,0) \cong \mathbb{CP}^2$; in general $M(d,n)$ is rational
- the action of $\mathrm{PGL}_2(\mathbb{C})$ on $Y_{d,n}^{ss}$ is free away from a locus of codimension ≥ 2 (unless $(d,n) = (2,0)$).

Due to this last point, *if $M(d,n)$ is $\mathbb{Q}$-factorial* then the pullback

$$\phi^* : \mathrm{Pic}(M(d,n)) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Pic}(Y_{d,n}^{ss}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Using methods adapted from [1] we were able to compute the space on the right. The **Picard group** is generated by the boundary divisors of $Y_{d,n}$ lying in the semistable locus together with $\mathrm{ev}_1^*(\mathcal{O}(1,0))$ for $n = 1,2$. Intersecting with explicit test curves we showed that all relations among these generators are pullbacks of boundary relations in $\overline{M}_{0,n}$ under the forgetful map $Y_{d,n} \to \overline{M}_{0,n}$.

Modular interpretation

The space M_d of degree d self maps and its compactification M_d^s have been studied before, see for instance [2]. There it was shown that M_d is a **coarse moduli space** for families $\mathbb{P}_S \to \mathbb{P}_S$ of degree d self-maps over S up to S-automorphisms of $\mathbb{P}_S$. We were able to show that the locus M_d^* of points with trivial $\mathrm{PGL}_2(\mathbb{C})$-isotropy is a **fine moduli space** parametrizing self maps of flat, projective families $\mathcal{C} \to S$ with geometric fibres $\mathbb{CP}^1$ that are not necessarily the trivial family $\mathbb{P}_S$. The corresponding universal family is the restricition of the forgetful morphism $M(d,1) \to M(d,0)$ to M_d^*.
As for the compactification, closed points of $M(d,n)$ are in bijection with stable degree d self-maps of n-pointed genus 0 curves (for a suitable definition of "stable self-map"). However, defining a corresponding modular property making $M(d,n)$ a coarse moduli space requires more care and is still work in progress.

Recursive boundary structure

Consider the boundary divisor $D_{B,k}$ of $M(d,n)$ of maps $f : C \to \mathbb{CP}^1 \times \mathbb{CP}^1$ where one component of C carries the markings $B \subset \{1, \ldots, n\}$ and maps with degree $(0,k)$. By removing this component and putting a new marking q at the intersection point with the other component, we obtain the graph of a degree $d-k$ self-map with $n-|B|+1$ markings. This leads to a generalization $M(d|d_1, \ldots, d_n)$ of $M(d,n)$ where markings can have a **weight** playing a role in the definition of semistability in the GIT quotient.

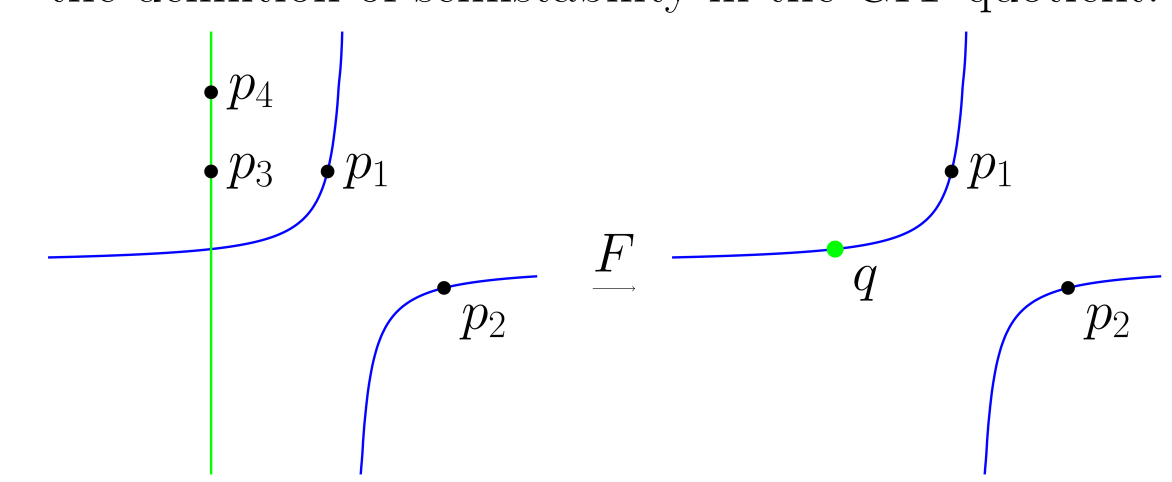

Figure 2: Forgetful map from divisor $D_{\{3,4\},1}$ in $M(2,4)$ to $M(1|0,0,1)$

The forgetful map F illustrated above exhibits the divisor $D_{B,k}$ as a fibration above $M(d-k|0,\ldots,0,k)$. This recursive structure might be used in exploring the intersection theory on the spaces $M(d|d_1,\ldots,d_n)$.

References

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Contact Information

- Email: johannes.schmitt@math.ethz.ch
- Phone: +41 44 632 3406