Graduate Seminar Advanced Topology (S4D4)

Higher algebra

Tuesdays, 14:15-15:45, SR 0.006

Stefan Schwede* & Jack Davies†

Two of the most salient features of singular cohomology are its multiplicative structure and Steenrod operations. These can be described in a homotopy theoretic manner by way of an $\mathbb{E}_\infty$*-ring structure* on the spectrum $\mathbb{Z}$, often also denoted as $H\mathbb{Z}$, representing singular cohomology. There are many other examples of $\mathbb{E}_\infty$-rings, coming from geometry, such as topological K-theory and bordism, from algebra, such as algebraic K-theory and (topological) Hochschild homology, and chromatic homotopy theory, such as Lubin–Tate theories and elliptic cohomology theories. The $\mathbb{E}_\infty$-structures on these spectra help us with specific computations and also provide a solid theoretical framework for higher algebra.

In this seminar, we will build up a language for $\mathbb{E}_\infty$-ring spectra and their ∞-categories of modules. Many formal analogies can be made to classical algebra, such as Morita theory and deformation theory, but there are also some subtle differences, such as the difficulty to construct $\mathbb{E}_\infty$-rings via generators and relations with manageable homotopy groups.

Each talk should be 90 minutes long, accounting for questions and comments, and so it is up to each presenter to chose exactly what should be presented from topic, although the main theorems and definitions should always be given. Summaries are below, although our main reference is Lurie's book [Lur17].

(08.10.2024) Presentable and stable ∞-categories (Carl Foth) Define an accessible ([Lur09b, §5.4]) and a (locally) presentable ([Lur09b, §5.5]) ∞-category and show that $\mathcal{A}$n and Sp are both presentable; see [Lur09b, Ex.5.5.1.8] and [Lur17, Pr.1.4.4.4], respectively. Prove the adjoint functor theorem for (locally) presentable ∞-categories; see [Lur09b, Cor.5.5.2.9]. Recall the definition of a stable ∞-category and show that Sp is the free presentable stable ∞-category on a single object denoted as $\mathbb{S}$; see [Lur17, Cor.1.4.4.6].

(15.10.2024) Stable ∞-categories and t-structures (David Bowman) Define a t-structure on a stable ∞-category ([Lur17, Df.1.2.1.1]) and accessible t-structure ([Lur17, Df.1.4.4.12]). Also discuss left and right completeness for t-structures and show the heart is a 1-category; see [Lur17, Rmk.1.2.1.12]. Show that Sp and the derived category $\mathcal{D}(R)$ of

*schwede@math.uni-bonn.de

†davies@math.uni-bonn.de

a ring R both have natural t-structures ([Lur17, Pr.1.4.3.4 & 1.3.2.19]) and calculate their hearts.

(22.10.2024) Symmetric monoidal ∞-categories (Semen Slobodianiuk) Describe the formulation of symmetric monoidal 1-categories in terms of op-fibrations over Fin_* [Lur17, §2] and introduce symmetric monoidal ∞-categories [Lur17, 2.0.0.7]. Give the examples of Cartesian monoidal structures [Lur17, 2.4.1.4] and nerves of symmetric monoidal model categories [NS18, Th.A.7].

(29.10.2024) Classical operads (Mathis Birken) Motivate operads with the Stasheff associahedra and the recognition theorem for loop spaces; see [Sta63]. Define operads in a symmetric monoidal 1-category as well as their algebras and modules over these algebras. Give examples such as the associative, commutative, Lie, endomorphism, and little cubes operads. Talk about the recognition theorem; see [BV73] and [May72]. Mention that coloured operads are a variant. Good references for all of this are [Lur17, §2] and [Fre09, §I], but most textbooks or papers discussing the foundations of operads would also suffice.

(05.11.2024) Algebras and modules over ∞-operads (Ben Steffan) Define ∞-operads ([Lur17, Df.2.1.1.10]) as well as their algebras ([Lur17, Df.2.1.3.1]) and modules ([Lur17, §3.3]). Give an overview of the proof of [Lur17, Th.3.4.4.2].

(12.11.2024) The Lurie tensor product of ∞-categories (Yining Chen) Define the tensor product of an ∞-category ([Lur17, §4.8.1]), how this gives a tensor product on Pr^L ([Lur17, Pr.4.8.1.15]), and how Sp can be endowed with a unique symmetric monoidal structure ([Lur17, Cor.4.8.2.19]).

(19.11.2024) Structured ring spectra (Julius Mann) Define $\mathbb{E}_1$- and $\mathbb{E}_\infty$-rings; see [Lur17, Df.4.1.1.6 & 7.1.0.1]. Give examples including the sphere, spherical monoid rings, and Eilenberg–Mac Lane spectra ([Lur17, Pr.7.1.3.18]), as well as general constructions such as localisations ([Lur17, Pr.7.2.3.27]) and completions ([Lur18, Cor.7.3.5.2]) if there is time; this might be better approached by showing the localisation functors $\mathrm{Sp} \to \mathrm{Sp}_E$, where $E = \mathbb{S}/p$ or $\mathbb{S}[\mathbb{P}^{-1}]$, are compatible with the $\mathcal{O}$-monoidal structure á la [Lur17, Df.2.2.1.6].

(26.11.2024) Modules over ring spectra (Emanuele Cortinovis) Define the ∞-category of modules LMod_R over an $\mathbb{E}_1$-ring R ([Lur17, Df.7.1.1.2]), show it is stable ([Lur17, Cor.7.1.1.5]) and has a t-structure ([Lur17, Pr.7.1.1.3]). Show that the category of modules over an $\mathbb{E}_\infty$-ring is again a symmetric monoidal ∞-category ([Lur17, §3.3.3 & Th.3.3.3.9]), define $\mathbb{E}_1$- and $\mathbb{E}_\infty$-objects in Mod_R ([Lur17, Df.7.1.3.5]), and discuss how if R is connective then the t-structure on Mod_R yields truncations and connective cover functors on Alg_R and CAlg_R; see [Lur17, Pr.7.1.3.13-15].

(03.12.2024) Morita theory (Congzheng Liu) Prove the ∞-categorical Schwede–Shipley theorem ([Lur17, Th.7.1.2.1]) as well as [Lur17, Pr.7.1.2.7]. Given some examples, including that $\mathcal{D}(R) \simeq \mathrm{Mod}_R(\mathrm{Sp})$ for a commutative ring R and that $\mathcal{D}(\mathbb{Q}) \simeq \mathrm{Sp}_{\mathbb{Q}}$; see [Lur17, Th.7.1.2.13].

(10.12.2024) General theory of the cotangent complex (Yiyang Chang) Define the tangent bundle of a presentable ∞-category $\mathcal{C}$ ([Lur17, Df.7.3.1.9]), relative adjunctions ([Lur17, Df.7.3.2.2]), and the relative cotangent complex functor ([Lur17, Df.7.3.3.1]). Discuss the proof of [Lur17, Th.7.3.4.13 & 7.3.4.18]. Also see [Lur09a, §1].

(17.12.2024) The cotangent complex of an $\mathbb{E}_\infty$-ring (Fabio Neugebauer) Specialise the discussion from last time to the case where $\mathcal{C} = \mathrm{CAlg}$. Discuss square-zero extensions of $\mathbb{E}_\infty$-rings ([Lur17, Df.7.4.1.6]) and [Lur17, Th.7.4.1.26] which leads to the fact that the Postnikov tower of an $\mathbb{E}_\infty$-ring is comprised of square-zero extensions ([Lur17, Cor.7.4.1.28]). Prove some of the basic finiteness and connectivity properties of the cotangent complex such as the consequences of [Lur17, Th.7.4.3.1], the relation to the module of differentials ([Lur17, Pr.7.4.3.9]), and the finiteness conditions of [Lur17, Th.7.4.3.18]. Also see [Lur09a, §2].

(07.01.2025) Lurie's étale rigidity (Bimit Mandal) Fix an $\mathbb{E}_\infty$-ring R. Show that the functor

$$\mathrm{CAlg}_R^{\text{ét}} \xrightarrow{\pi_0} \mathrm{CAlg}_{\pi_0 R}(\mathrm{Ab})^{\text{ét}}$$

is an equivalence of ∞-categories; see either [Lur09a, Th.3.4.1] or [Lur17, Th.7.5.4.2]. Motivate this with a discussion of the nonexistence of a *Gaussian sphere* $\mathbb{S}[i]$ as an $\mathbb{E}_\infty$-ring following [SVW99]. Give some more examples of étale realisations generalising localisation results from the "structured ring spectra" talk.

References

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347 of *Lect. Notes Math.* Springer, Cham, 1973.

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[Lur17] Jacob Lurie. Higher algebra. Available at `https://www.math.ias.edu/~lurie/`, 2017.

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[Sta63] James D. Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Am. Math. Soc.*, 108:275–292, 293–312, 1963.

[SVW99] R. Schwänzl, R. M. Vogt, and F. Waldhausen. Adjoining roots of unity to E_∞ ring spectra in good cases – a remark. In *Homotopy invariant algebraic structures. A conference in honor of J. Michael Boardman. AMS special session on homotopy theory, Baltimore, MD, USA, January 7–10, 1998*, pages 245–249. Providence, RI: American Mathematical Society, 1999.