CHARACTER THEORY AND EULER CHARACTERISTIC FOR ORBISPACES AND INFINITE GROUPS

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Abstract. Given a discrete group G with a finite model for $\underline{E}G$, we study $K(n)^*(BG)$ and $E^*(BG)$, where $K(n)$ is the n-th Morava K-theory for a given prime and E is the height n Morava E-theory. In particular we generalize the character theory of Hopkins, Kuhn and Ravenel who studied these objects for finite groups. We give a formula for a localization of $E^*(BG)$ and the $K(n)$-theoretic Euler characteristic of BG in terms of centralizers. In certain cases these calculations lead to a full computation of $E^*(BG)$, for example when G is a right angled Coxeter group, and for $G = SL_3(\mathbb{Z})$. We apply our results to the mapping class group $\Gamma_{\frac{p-1}{2}}$ for an odd prime p and to certain arithmetic groups, including the symplectic group $Sp_{p-1}(\mathbb{Z})$ for an odd prime p and $SL_2(\mathcal{O}_K)$ for a totally real field K.

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Introduction

The purpose of this paper is to extend the generalized character theory of Hopkins, Kuhn and Ravenel [HKR00] from finite groups to discrete groups G that admit a finite model for the classifying space of proper actions. We demonstrate the usefulness of our generalization by explicitly calculating the Morava K-theory Euler characteristic and the E-cohomology of BG for several interesting examples of such groups that arise in geometric group theory and arithmetic, including the

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mapping class group $\Gamma_{\frac{p-1}{2}}$ for an odd prime p, the symplectic group $Sp_{p-1}(\mathbb{Z})$ for $p \geq 5$, and $SL_2(\mathcal{O}_K)$ for a totally real field K.

To put our results into context, we briefly review some of the highlights of [HKR00]. The Atiyah–Segal completion theorem [AS69] identifies the complex K-theory of the classifying space of a finite group G with the completion of the complex representation ring $R(G)$ at its augmentation ideal. And classical character theory identifies $R(G) \otimes \mathbb{C}$ with the ring $\mathrm{Cl}(G, \mathbb{C})$ of complex valued class functions on G. Hopkins, Kuhn and Ravenel generalize these ideas to higher chromatic heights: in [HKR00], they study the generalized cohomology ring $E^*(BG)$ for finite groups G and for specific complex oriented cohomology theories E that generalize p-adic complex K-theory.

Let $K(n)$ denote the n-th Morava K-theory at a prime p, with the coefficient ring $K(n)^* = \mathbb{F}_p[v_n^{\pm 1}]$, where v_n is of degree $-2(p^n - 1)$. A result of Ravenel [Rav82] shows that for G finite, $K(n)^*(BG)$ is a finitely generated graded vector space over the evenly graded field $K(n)^*$. Hence it is meaningful to define the $K(n)$-theory Euler characteristic as

$$\chi_{K(n)}(BG) = \dim_{K(n)^*} K(n)^{\mathrm{even}}(BG) - \dim_{K(n)^*} K(n)^{\mathrm{odd}}(BG).$$

One of the main results of Hopkins, Kuhn and Ravenel computes $\chi_{K(n)}(BG)$ in group-theoretic terms. The group G acts by simultaneous conjugation on the set $G_{n,p}$ of n-tuples of pairwise commuting elements of p-power order. By [HKR00, Theorem B], the Euler characteristic $\chi_{K(n)}(BG)$ equals the number of G-orbits of $G_{n,p}$. A particularly interesting special case is when $K(n)^*(BG)$ is concentrated in even degrees; then the formula of Hopkins, Kuhn and Ravenel computes the dimension of $K(n)^*(BG)$ as a graded $K(n)^*$-module.

The importance of this computation becomes clear when trying to understand $E^*(BG)$, where E is the Morava E-theory spectrum at the height n and prime p. Hopkins, Kuhn and Ravenel introduce a ring $L(E^*)$ which is faithfully flat over $p^{-1}E^*$, and hence a rational algebra. Roughly speaking, $L(E^*)$ can be thought of as a character ring obtained by adjoining enough p-th power roots of unity to E^*. In [HKR00, Theorem C], Hopkins, Kuhn and Ravenel exhibit an isomorphism

$$L(E^*) \otimes_{E^*} E^*(BG) \cong \mathrm{Cl}_{n,p}(G; L(E^*))$$

to the ring of 'generalized class functions', i.e., functions from $G_{n,p}$ to $L(E^*)$ that are constant on orbits of the conjugation action by G. These results opened the door to an integral understanding of $E^*(BG)$ for finite groups G. Since $E^*(BG)$ is finitely generated as a graded E^*-module by [GS99, Corollary 4.4], a result of Strickland [Str98] implies that if $K(n)^*(BG)$ is even, then $E^*(BG)$ is even and free as a graded E^*-module. Hence the above rational computation of $E^*(BG)$ leads to the integral computation.

Our results. We generalize the results of Hopkins, Kuhn and Ravenel from finite groups to infinite discrete groups with a finite model for $\underline{E}G$, the classifying G-space for proper actions. Many interesting groups in geometric group theory and arithmetic fall into this class, and often important moduli spaces are modeled by the quotient space $\underline{B}G = G\backslash\underline{E}G$, see for example [Lüc05b]. Our main motivation is to study new invariants for such groups, and to see how much chromatic homotopy theory is reflected in the geometric group theory and number theory. Another motivation is the work of the first author on Chern characters and rational equivariant cohomology theories. The papers [Lüc02a, Lüc07, Lüc05a] exhibit formulas for such cohomology theories, and in particular for the rational K-theory, of the classifying space BG, which are very similar to the formulas of Hopkins–Kuhn–Ravenel. Analogous formulas are also provided by Adem [Ade92] in the case of virtually torsion free groups. All these formulas generalize the splitting for the rational equivariant K-theory which goes back to Artin, tom Dieck and Atiyah–Segal. The present paper is a common generalization of [HKR00] and [Ade92, Lüc02a, Lüc07, Lüc05a]: On the one hand we extend the results of [HKR00] from finite groups to certain infinite groups, and on the other hand we generalize the results of [Ade92, Lüc02a, Lüc07, Lüc05a] from the chromatic height 1 to arbitrary chromatic height.

Our first main result, proved as Corollary 2.5, generalizes the group-theoretic formula for the Morava K-theory Euler characteristic [HKR00, Theorem B] to infinite groups.

Theorem A. *Let G be a discrete group that admits a finite G-CW-model for $\underline{E}G$. Then*

$$\chi_{K(n)}(BG) = \sum_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} \chi_{\mathbb{Q}}(BC\langle g_1,\ldots,g_n\rangle).$$

Here $G_{n,p}$ denotes the set of n-tuples of pairwise commuting elements of G, each of which has p-power order. The group G acts on $G_{n,p}$ by simultaneous conjugation. Further, $C\langle g_1,\ldots,g_n\rangle$ is the centralizer of the abelian subgroup $\langle g_1,\ldots,g_n\rangle$ generated by $g_1, g_2, \ldots, g_n$, and $\chi_{\mathbb{Q}}$ denotes the rational homology Euler characteristic. Since G admits a finite model for $\underline{E}G$, the set $G\backslash G_{n,p}$ of orbits is finite.

In Theorem A, a new feature shows up for infinite discrete groups that is not present in the work of Hopkins–Kuhn–Ravenel. If G is finite, then so are the centralizers $C\langle g_1,\ldots,g_n\rangle$, so their classifying spaces are rationally acyclic, and $\chi_{\mathbb{Q}}(BC\langle g_1,\ldots,g_n\rangle) = 1$. Hence for finite groups, the right hand side of the formula in Theorem A is a fancy way to write the number of G-orbits on $G_{n,p}$. For infinite discrete groups, in contrast, the centralizers occurring in Theorem A need not be finite nor rationally acyclic. So the rational Euler characteristic $\chi_{\mathbb{Q}}(BC\langle g_1,\ldots,g_n\rangle)$ need not be 1, and the right hand side in Theorem A reflects combinatorial properties of the group G. In Example 2.10, we illustrate this phenomenon with the amalgamated product $G = D_8 *_{C_4} D_8$ of two copies of the dihedral group of order 8 along the cyclic group of order 4. One of the centralizers in this case is isomorphic to $C_4 \times \mathbb{Z}$ and hence not rationally acyclic.

Our second main result, proved as Corollary 3.5, generalizes the character isomorphism [HKR00, Theorem C] to infinite groups.

Theorem B. *Let G be a discrete group that admits a finite G-CW-model for $\underline{E}G$. Then the character map*

$$\chi_{n,p} \;:\; L(E^*)\otimes_{E^*} E^*(BG) \xrightarrow{\cong} \prod_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} H^*(BC\langle g_1,\ldots,g_n\rangle; L(E^*))$$

is an isomorphism.

Again, the situation is more subtle than for finite groups. If G is finite, then $BC\langle g_1,\ldots,g_n\rangle$ is rationally acyclic, and hence $L(E^*)$-acyclic. So for finite groups, the right hand side of the character isomorphism is a free $L(E^*)$-module whose rank is the number of G-orbits on $G_{n,p}$, and thus isomorphic to the generalized class functions $\mathrm{Cl}_{n,p}(G; L(E^*))$. As already mentioned above, for infinite discrete groups, the centralizers $C\langle g_1,\ldots,g_n\rangle$ need not be finite nor rationally acyclic. So the factors on the right hand side of the character isomorphism need not simply be free $L(E^*)$-modules of rank 1. In the special case of chromatic height 1, the theory E is the p-adic K-theory and $L(E^*) = \mathbb{Q}_p(\zeta_{p^\infty})[u^{\pm 1}]$, where u is the Bott class. Hence for $n = 1$, Theorem B recovers a p-adic version of [Lüc07, Theorem 0.1] and the second formula in [Ade92, Theorem 4.2].

Our third main result relates the Morava K-theory Euler characteristic of an infinite group to its orbifold Euler characteristic. The latter is defined for virtually torsion-free groups G with finite virtual cohomological dimension, as follows: If Γ is any finite index torsion-free subgroup of G, then

$$\chi_{\mathrm{orb}}(BG) = \chi_{\mathbb{Q}}(B\Gamma)/[G:\Gamma],$$

compare [Wal61, Ser71]; here $\chi_{\mathbb{Q}}$ is again the rational homology Euler characteristic, which coincides with the classical Euler characteristic for a finite complex. This rational number often reflects geometric and arithmetic properties of the group G. A good example is provided by the mapping class group Γ_g^1 of a closed orientable surface of genus $g \geq 1$ with one marked point. By a theorem of Harer and Zagier [HZ86], the orbifold Euler characteristic $\chi_{\mathrm{orb}}(B\Gamma_g^1)$ equals the special value $\zeta(1-2g)$ of the Riemann ζ-function.

In Theorem 4.9 we establish the following relation:

Theorem C. *Let G be a discrete group that admits a finite G-CW-model for $\underline{E}G$, and such that every torsion element in G has p-power order. Then*

$$\chi_{K(n)}(BG) = \sum_{[g_0,g_1,\dots,g_n]\in G\backslash G_{n+1,p}} \chi_{\mathrm{orb}}(BC\langle g_0,g_1,\dots,g_n\rangle).$$

As we explain in more detail in Remark 4.10, these results motivate that for discrete groups with only p-power torsion for a fixed prime p, the orbifold Euler characteristic can be thought as the chromatic height -1 invariant. Indeed, we get a chromatic sequence starting at 'height -1':

$$\chi_{\mathrm{orb}}, \chi_{\mathbb{Q}}, \chi_{K(1)}, \dots, \chi_{K(n)}, \dots,$$

where the i-th term can be obtained from the j-th term for $j < i$ using the tuples of $i-j$ many commuting elements of finite order. In Remark 4.10 we also explain how this result is related to the paper of Yanovski [Yan23].

Computations and applications. The long final Section 5 discusses several classes of infinite groups G with finite model for $\underline{E}G$, and it illustrates our general theory with concrete computations. We fully compute Morava K-theory and E-theory for right angled Coxeter groups and the general linear group $SL_3(\mathbb{Z})$. Computations for the right angled Coxeter groups generalize the results of [DL17] to higher chromatic heights. The computations for $SL_3(\mathbb{Z})$ partially recover the results of [TY92]. Further, we compute $\chi_{K(n)}(BG)$ and $L(E^*)\otimes_{E^*} E^*(BG)$ for G a crystallographic group of the form $\mathbb{Z}^n \rtimes \mathbb{Z}/p$. At an odd prime p, we also give explicit formulas for the mapping class group $\Gamma_{\frac{p-1}{2}}$ of a closed oriented surface of genus $\frac{p-1}{2}$, for $G = SL_2(O_K)$, where K is a totally real field, and for $G = GL_{p-1}(\mathbb{Z}), SL_{p-1}(\mathbb{Z}), Sp_{p-1}(\mathbb{Z})$. These formulas involve explicit number theoretic invariants such as special values of zeta functions and class numbers. We list some of them here; the results are all new for the height $n > 1$, and some also for $n = 1$:

- For every totally real field K and odd prime p,

$$\chi_{K(n)}(BSL_2(\mathcal{O}_K)) = 2\zeta_K(-1) + \sum_{(H)} (|H_{(p)}|^n - \tfrac{2}{|H|}),$$

  where ζ_K is the Dedekind zeta function of K, see Proposition 5.17. The sum runs over the conjugacy classes of the maximal finite subgroups (which are all abelian). The formula generalizes Brown's formula [Bro74, Section 9.1] for $\chi_{\mathbb{Q}}(BSL_2(\mathcal{O}_K))$ to higher chromatic heights. The formula was already known for $n = 1$ case, see [Ade92, Example 4.4].

- For $p \geq 5$ and $G = GL_{p-1}(\mathbb{Z})$, we get the identity $\chi_{K(n)}(BGL_{p-1}(\mathbb{Z})) = \chi_{\mathbb{Q}}(BGL_{p-1}(\mathbb{Z}))$, see Proposition 5.21. The invariant $\chi_{\mathbb{Q}}(BGL_{p-1}(\mathbb{Z}))$ is known to vanish for $p \geq 13$ [Hor05, Theorem 0.1 (a)]. However, for the height n Morava E-theory at the prime p, the character theory still provides an interesting formula
$$L(E^*) \otimes_{E^*} E^*(BGL_{p-1}(\mathbb{Z})) \cong \\ H^*(BGL_{p-1}(\mathbb{Z}); L(E^*)) \oplus \bigoplus_{\frac{p^n-1}{p-1} |\operatorname{Cl}(\mathbb{Q}(\zeta_p))|} (L(E^*) \oplus L(E^*)[1])^{\otimes_{L(E^*)} \frac{p-3}{2}},$$
where $|\operatorname{Cl}(\mathbb{Q}(\zeta_p))|$ is the class number of the cyclotomic field $\mathbb{Q}(\zeta_p)$. The rational cohomology $H^*(BGL_{p-1}(\mathbb{Z}); \mathbb{Q})$ is not known in general. However it is known for $p = 3, 5, 7$, so that for these primes we can compute $L(E^*) \otimes_{E^*} E^*(BGL_{p-1}(\mathbb{Z}))$ as a graded $L(E^*)$-module; see Remark 5.24 for the case $p = 3$. Again the case $n = 1$ was already considered in [Ade92, Example 4.5].
- For any odd prime p, we establish the formula
$$\chi_{K(n)}(BSp_{p-1}(\mathbb{Z})) = \chi_{\mathbb{Q}}(BSp_{p-1}(\mathbb{Z})) + 2^{\frac{p-1}{2}} h_p^- \cdot \tfrac{p^n-1}{p-1},$$
where h_p^- is the relative class number of the cyclotomic field $\mathbb{Q}(\zeta_p)$, see Proposition 5.28.
- For a prime $p \geq 5$, we derive the formula
$$\chi_{K(n)}(B\Gamma_{\frac{p-1}{2}}) = \chi_{\mathbb{Q}}(B\Gamma_{\frac{p-1}{2}}) + \tfrac{(p^n-1)(p+1)}{6},$$
see Proposition 5.30.

Orbispaces. At the technical level, a key tool is the systematic use of orbispaces in the sense of [GH07]. We generalize several aspects of the generalized character theory of Hopkins, Kuhn and Ravenel from the class of finite G-CW-complexes for finite groups G to the class of compact orbispaces. The connection to geometric group theory stems from the fact that compact orbispaces include the global classifying spaces of all infinite discrete groups with finite G-CW-model for $\underline{E}G$. More generally for any discrete group G and finite proper G-CW complex A, the global quotient orbispace $G\backslash\!\!\backslash A$ is compact. Similar generalizations were investigated by Lurie in [Lur19], though applications to infinite groups have not been explored there. See the remark below at the end of the introduction.

The class of compact orbispaces provides a unified approach to the different kinds of Euler characteristics relevant for this paper. The relationships between these invariants are conceptually explained by explicit constructions in the category of orbispaces, such as the formal loop space introduced in Construction 4.3. However, for the readers who prefer to avoid orbispaces we provide Remarks 2.8 and 3.6, where alternative ways of proving the above theorems are sketched. These approaches use equivariant cohomology theories and G-CW decompositions along the lines of [Lüc02b, Lüc05a], and might be more accessible for the readers working in geometric group theory.

Organization. Here is a more detailed outline of the contents of this paper. In Section 1 we recall generalities on orbispaces. The global classifying space $B_{\mathrm{gl}}G$ of a discrete group G is introduced in Example 1.5. We recall the concept of *compact* orbispace and record some finiteness results in Theorem 1.12. An easy but fundamental fact is that when G admits a finite model for $\underline{E}G$, then its global classifying space $B_{\mathrm{gl}}G$ is a compact orbispace. In Theorem 1.16 we calculate the orbispace Burnside ring A_{orb}, the universal recipient for additive invariants of compact orbispaces; this calculation is used in Definition 1.19 to define the orbispace Euler characteristic $\chi_{\mathrm{orb}}[X]$. This invariant generalizes the orbifold Euler characteristic of virtually torsion-free groups with finite $\underline{E}G$, by the relation $\chi_{\mathrm{orb}}(BG) = \chi_{\mathrm{orb}}[B_{\mathrm{gl}}G]$.

Section 2 studies the Morava K-theory Euler characteristic of compact orbispaces and discrete groups, possibly infinite. The main result is Theorem 2.4; for a compact orbispace X, it identifies the Morava K-theory Euler characteristic of the underlying space as the rational Euler characteristic of a certain space $X\langle\mathbb{Z}_p^n\rangle$ made from the values of X at all finite quotients of $\mathbb{Z}_p^n$. This result generalizes the Hopkins–Kuhn–Ravenel formula $\chi_{K(n)}(BG) = |G\backslash G_{n,p}|$ for finite groups G, which is used as an input in the proof. Inspired by the work of Stapleton [Sta13], we also observe in Theorem 2.4 that for every compact orbispaces X and $m, n \geq 1$, the $K(m+n)$-Euler characteristic of the underlying space of X agrees with the $K(m)$-Euler characteristic of the space $X\langle\mathbb{Z}_p^n\rangle$. In Corollary 2.5 we specialize the theorem to global quotient orbispaces. When applied to $B_{\mathrm{gl}}G$ for discrete groups G with finite model for $\underline{E}G$, it yields our Theorem A, as well as the identity

$$\chi_{K(m+n)}(BG) = \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} \chi_{K(m)}(BC\langle g_1,\dots,g_n\rangle)$$

for all $m, n \geq 1$.

Section 3 generalizes the character theory of Hopkins–Kuhn–Ravenel. In Construction 3.1, we extend their character map from global quotients of finite group actions to compact orbispaces, and we show in Theorem 3.4 that it is an isomorphism. When applied to $B_{\mathrm{gl}}G$ for discrete groups with finite model for $\underline{E}G$, it yields our Theorem B, compare Corollary 3.5.

In Section 4 we relate the orbifold Euler characteristic to the Morava K-theoretic Euler characteristic. The upshot is that $\chi_{\mathrm{orb}}(BG)$ displays features of a chromatic height -1 invariant, see Remark 4.10. For this discussion we make use of the *formal loop space* $\mathcal{L}X$ of an orbispace X, see Construction 4.3. We show in Theorem 4.7 that for a compact orbispace, the formal loop orbispace is again compact, and the orbispace Euler characteristic of $\mathcal{L}X$ equals the rational Euler characteristic of the underlying space of X. We already gave an outline of Section 5, which discusses many examples.

Related work. Many results in this paper are related to the formulas in [Sta13]. Though the paper [Sta13] deals with finite groups, the idea of using the full cohomology of centralizers in this context is already present there. Similar ideas in the context of $\mathbb{F}_p$-cohomology are present in [Lee94], [Hen97], [Hen99] and [JM92]. We expect that the results in this paper can be generalized to the transchromatic context and many results in [Sta13] have versions for infinite discrete groups.

Lurie generalizes Hopkins–Kuhn–Ravenel character theory to the case of orbispaces in [Lur19], see also an exposition in [Rak15]. The approach in [Lur19] is slightly different from ours since Lurie uses global spaces defined on finite abelian groups (which he also calls 'orbispaces'). This allows him to reproduce the results of [HKR00] and [Sta13] and generalize them to the context of orbispaces. We are interested in exploring the application of the character theory in the context of infinite groups and geometric group theory. For this purpose orbispaces seem to provide a very convenient language. This is why our orbispaces are directly defined on all finite groups as opposed to finite abelian groups as considered in [Lur19].

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1. Orbispaces

In this section we recall some basic facts about orbispaces in the sense of Gepner and Henriques [GH07]. Important examples of orbispaces are global classifying spaces of discrete groups, and more generally global quotient spaces associated to G-spaces, see Example 1.5. We recall the concept of *compact* orbispace and record some finiteness results in Theorem 1.12. An easy but fundamental fact is that when G admits a finite model for $\underline{E}G$, then its global classifying space $B_{\mathrm{gl}}G$ is a compact orbispace.

Then we introduce the *orbispace Burnside ring* A_{orb}, the universal recipient of an Euler characteristic for compact orbispaces. We also show that the underlying abelian group of the orbispace Burnside ring is free, with a basis parameterized by isomorphism classes of finite groups. Altogether, this means that in order to specify an Euler characteristic with values in some abelian group, we can freely assign values to the global classifying spaces of finite groups. In Definition 1.19 we define the orbispace Euler characteristic $\chi_{\mathrm{orb}}[X]$ for compact orbispaces. This invariant generalizes the orbifold Euler characteristic of virtually torsion-free groups with finite $\underline{E}G$, by the relation $\chi_{\mathrm{orb}}(BG) = \chi_{\mathrm{orb}}[B_{\mathrm{gl}}G]$.

Construction 1.1 (The global indexing categories)**.** We recall from [GH07] the topological category Glo and its non-full subcategory Orb. To distinguish the presheaves of spaces on Glo and Orb we adopt the terminology suggested by Rezk [Rez14] and use the terms *global spaces* and *orbispaces*, respectively. The reader should beware, however, that global spaces are also called orbispaces by some authors, for example in [GH07, Lur19]. For this paper, orbispaces are the more important class, but global spaces show up in the discussion of the formal loop space in Section 4.

In short, the topological category Glo is the coherent nerve of the $(2,1)$-category of groups, group homomorphisms, and conjugations. And Orb is the coherent nerve of the $(2,1)$-category of groups, injective group homomorphisms, and conjugations. We expand this definition. The objects of Glo and Orb are all discrete groups. If G and K are two groups, the morphism space $\mathrm{Glo}(K,G)$ is the geometric realization of the translation groupoid $\underline{\mathrm{Hom}}(K,G)$ of the conjugation G-action on $\mathrm{Hom}(K,G)$. Unpacking this even further, the groupoid $\underline{\mathrm{Hom}}(K,G)$ has group homomorphisms $\alpha : K \longrightarrow G$ as objects, and morphisms are pairs (g,α) in $G \times \mathrm{Hom}(K,G)$, whose source and target are α and $c_g \circ \alpha$, respectively, with $c_g(\gamma) = g\gamma g^{-1}$ the inner automorphism. We thus have a homeomorphism

$$\mathrm{Glo}(K,G) \cong EG \times_G \mathrm{Hom}(K,G),$$

where EG is the bar construction model for the universal free G-space, and the right hand side is the homotopy orbit space of the G-action, by conjugation, on the set of group homomorphisms. The space $\mathrm{Glo}(K,G)$ is thus a 1-type, and there is a homotopy equivalence

$$\mathrm{Glo}(K,G) \simeq \coprod_{[\alpha:K\longrightarrow G]} BC(\alpha).$$

Here the disjoint union is over conjugacy classes of homomorphisms from K to G; and $C(\alpha)$ is the centralizer, in G, of the image of $\alpha : K \longrightarrow G$. Composition in Glo arises as geometric realization of the composition functor

$$\circ \ : \ \underline{\mathrm{Hom}}(K,G) \times \underline{\mathrm{Hom}}(L,K) \longrightarrow \underline{\mathrm{Hom}}(L,G) \tag{1.2}$$

that is composition of group homomorphisms on objects, and

$$(g,\alpha)\circ(k,\beta) = (g\cdot\alpha(k), \alpha\circ\beta)$$

on morphisms. Then Orb is the non-full topological subcategory of Glo spanned by the monomorphisms. In other words, $\mathrm{Orb}(K,G)$ is the union of those path components of $\mathrm{Glo}(K,G)$ that are indexed by injective group homomorphisms. Thus

$$\mathrm{Orb}(K,G) \cong EG\times_G \mathrm{Mono}(K,G) \simeq \coprod_{[\alpha:K\longrightarrow G]} BC(\alpha), \tag{1.3}$$

where this time, the disjoint union is over G-conjugacy classes of monomorphisms from K to G.

We write $\mathrm{Orb}_{\mathcal{F}in}$ for the full topological subcategory of Orb spanned by the finite groups.

Definition 1.4. An *orbispace* is a contravariant continuous functor from $\mathrm{Orb}_{\mathcal{F}in}$ to the category $\mathbf{T}$ of compactly generated spaces. The category of orbispaces is denoted by *orbspc*. A morphism $f : X \longrightarrow Y$ of orbispaces is an *equivalence* if for every finite group K, the map $f(K) : X(K) \longrightarrow Y(K)$ is a weak homotopy equivalence.

For this paper, the most important examples of orbispaces are global quotients of actions of discrete groups.

Example 1.5 (Global quotient orbispaces)**.** We let G be a discrete group, and we let A be a G-space. The *global quotient orbispace* $G\backslash\!\!\backslash A$ is defined as follows. For a finite group K, we let G act on the space

$$\coprod_{\alpha\in\mathrm{Mono}(K,G)} A^{\mathrm{Im}(\alpha)}$$

by $g\cdot(a,\alpha) = (ga, c_g\circ\alpha)$. Then we define $(G\backslash\!\!\backslash A)(K)$ as the homotopy orbit space

$$(G\backslash\!\!\backslash A)(K) \ = \ EG\times_G \left(\coprod_{\alpha\in\mathrm{Mono}(K,G)} A^{\mathrm{Im}(\alpha)}\right).$$

In particular, the underlying space $(G\backslash\!\!\backslash A)(1)$ is the homotopy orbit space $EG\times_G A$. Choosing representatives of the conjugacy classes of monomorphisms $K \longrightarrow G$ exhibits the homotopy type as

$$(G\backslash\!\!\backslash A)(K) \ \simeq \ \coprod_{[\alpha]\in G\backslash\,\mathrm{Mono}(K,G)} EC(\alpha)\times_{C(\alpha)} A^{\mathrm{Im}(\alpha)},$$

where the square brackets indicate conjugacy classes.

To define the functoriality of $G\backslash\!\!\backslash A$ we let L be another finite group. We reinterpret $(G\backslash\!\!\backslash A)(K)$ a as homotopy colimit, and derive the Orb-functoriality from the functoriality of homotopy colimits. We write $\underline{\mathrm{Mono}}(K,G)$ for the translation category of the conjugation G-action on $\mathrm{Mono}(K,G)$. In other words, $\underline{\mathrm{Mono}}(K,G)$ is the full subcategory of the translation category $\underline{\mathrm{Hom}}(K,G)$ discussed in Construction 1.1, with objects the injective homomorphisms. The space $\mathrm{Orb}(K,G)$ is then the nerve of the category $\underline{\mathrm{Mono}}(K,G)$. The composition functors (1.2) restrict to monomorphisms, and make $\underline{\mathrm{Mono}}$ a sub-2-category of the $(2,1)$-category $\underline{\mathrm{Hom}}$.

The G-space A gives rise to a functor

$$A\sharp K \ : \ \underline{\mathrm{Mono}}(K,G) \ \longrightarrow \ \mathbf{T}$$

to the category of spaces that sends a monomorphism $\alpha : K \longrightarrow G$ to $A^{\mathrm{Im}(\alpha)}$, and a morphism (g,α) of $\underline{\mathrm{Mono}}(K,G)$ to the translation map $g\cdot : A^{\mathrm{Im}(\alpha)} \longrightarrow A^{\mathrm{Im}(c_g\circ\alpha)}$. The space $(G\backslash\!\backslash A)(K)$ is then precisely the homotopy colimit, in the sense of Bousfield–Kan [BK72, Chapter XII], of the functor $A\sharp K$. For every object (α,β) of $\underline{\mathrm{Mono}}(K,G)\times\underline{\mathrm{Mono}}(L,K)$, the group $\mathrm{Im}(\alpha\beta)$ is a subgroup of $\mathrm{Im}(\alpha)$, so the inclusion is a continuous map

$$(A\sharp K)(\alpha) \;=\; A^{\mathrm{Im}(\alpha)} \;\longrightarrow\; A^{\mathrm{Im}(\alpha\beta)} \;=\; (A\sharp L)(\alpha\circ\beta)\ .$$

For all morphisms $((g,\alpha),(k,\beta))$ of $\underline{\mathrm{Mono}}(K,G)\times\underline{\mathrm{Mono}}(L,K)$, the following square commutes:

$$\begin{tikzcd}
A^{\mathrm{Im}(\alpha)} \arrow[r, "\text{inclusion}"] \arrow[d, "g\cdot"'] & A^{\mathrm{Im}(\alpha\beta)} \arrow[d, "g\alpha(k)\cdot"] \\
A^{\mathrm{Im}(c_g\alpha)} \arrow[r, "\text{inclusion}"'] & A^{\mathrm{Im}(c_g\alpha c_k\beta)}
\end{tikzcd}$$

This means that the inclusions define a natural transformation of functors from the left-lower composite to the upper-right composite in the following diagram of functors:

$$\begin{tikzcd}
\underline{\mathrm{Mono}}(K,G)\times\underline{\mathrm{Mono}}(L,K) \arrow[r, "\text{composition}"] \arrow[d, "\text{projection}"'] & \underline{\mathrm{Mono}}(L,G) \arrow[d, "A\sharp L"] \\
\underline{\mathrm{Mono}}(K,G) \arrow[r, "A\sharp K"'] \arrow[ur, phantom, "\overset{\text{inclusion}}{\Rightarrow}"] & \mathbf{T}
\end{tikzcd}$$

This natural transformation then induces a continuous map of Bousfield–Kan homotopy colimits, and thus a continuous map

$$\begin{aligned}
(G\backslash\!\backslash A)(K)\times \mathrm{Orb}(L,K) = &\\
&\mathrm{hocolim}_{\underline{\mathrm{Mono}}(K,G)\times\underline{\mathrm{Mono}}(L,K)}(A\sharp K)\circ \text{projection}\\
\xrightarrow{\text{inclusion}_*}\ &\mathrm{hocolim}_{\underline{\mathrm{Mono}}(K,G)\times\underline{\mathrm{Mono}}(L,K)}(A\sharp L)\circ \text{composition}\\
\longrightarrow\ &\mathrm{hocolim}_{\underline{\mathrm{Mono}}(L,G)}(A\sharp L) = (G\backslash\!\backslash A)(L)
\end{aligned}$$

This defines the Orb-functoriality of $G\backslash\!\backslash A$.

As the following proposition shows, the orbispace $G\backslash\!\backslash A$ only sees the 'proper' G-homotopy type of A, and it is agnostic about the fixed point spaces for infinite subgroups of G. Said differently, the $G\backslash\!\backslash-$ is fully homotopical for '$\mathcal{F}in$-weak equivalences', i.e., weak equivalences on fixed points of all finite subgroups.

Proposition 1.6. *Let G be a discrete group. Let $f : A \longrightarrow B$ be a continuous G-map between G-spaces such that for every finite subgroup H of G, the map of fixed points $f^H : A^H \longrightarrow B^H$ is a weak equivalence. Then the morphism $G\backslash\!\backslash f : G\backslash\!\backslash A \longrightarrow G\backslash\!\backslash B$ is an equivalence of orbispaces.*

Proof. We let K be any finite group. Then for every monomorphism $\alpha : K \longrightarrow G$, the map $f^{\mathrm{Im}\,\alpha} : A^{\mathrm{Im}\,\alpha} \longrightarrow B^{\mathrm{Im}\,\alpha}$ is a weak equivalence by hypothesis. So the G-equivariant map

$$\coprod_{\alpha\in\mathrm{Mono}(K,G)} f^{\mathrm{Im}(\alpha)} : \coprod_{\alpha\in\mathrm{Mono}(K,G)} A^{\mathrm{Im}(\alpha)} \longrightarrow \coprod_{\alpha\in\mathrm{Mono}(K,G)} B^{\mathrm{Im}(\alpha)}$$

is a weak equivalence of underlying spaces. The homotopy orbit construction then takes it to a weak equivalence of spaces. So the morphism $G\backslash\!\backslash f$ is a weak equivalence at K. $\square$

Example 1.7 (Global classifying spaces)**.** The *global classifying space* of a discrete group G is the represented orbispace $B_{\mathrm{gl}}G = \mathrm{Orb}(-, G)$, or rather its restriction to the subcategory $\mathrm{Orb}_{\mathcal{F}in}$. By inspection of definitions, this is isomorphic to the global quotient orbispace of G acting on a one-point space, i.e.,

$$B_{\mathrm{gl}}G \ = \ G\backslash\!\backslash * \, .$$

By Proposition 1.6, the unique map $\underline{E}G \longrightarrow *$ induces an equivalence of orbispaces

$$G\backslash\!\backslash \underline{E}G \xrightarrow{\simeq} G\backslash\!\backslash * = B_{\mathrm{gl}}G.$$

Example 1.8. If G is a torsion-free group, then a monomorphism from a finite group to G must necessarily have a trivial source. So for every G-space A, the orbispace $G\backslash\!\backslash A$ has the homotopy orbit space $EG \times_G A$ as its value at the trivial group, and $G\backslash\!\backslash A$ is empty at all non-trivial finite groups.

Example 1.9. We let H be a subgroup of a discrete group G, and we let A be an H-space. We shall now specify an equivalence of orbispaces

$$H\backslash\!\backslash A \ \xrightarrow{\simeq} \ G\backslash\!\backslash (G \times_H A).$$

When A consists of a single point, this specializes to an equivalence from $B_{\mathrm{gl}}H$ to $G\backslash\!\backslash(G/H)$. For a finite group K, the inclusion $\iota : H \longrightarrow G$ induces a homotopy equivalence

$$(H\backslash\!\backslash A)(K) = EH \times_H \big(\coprod_{\beta \in \mathrm{Mono}(K,H)} A^{\mathrm{Im}(\beta)}\big) \xrightarrow{\simeq} EG \times_H \big(\coprod_{\beta \in \mathrm{Mono}(K,H)} A^{\mathrm{Im}(\beta)}\big).$$

Moreover, the G-equivariant homeomorphism

$$\begin{array}{ccc} G \times_H \big(\coprod_{\beta \in \mathrm{Mono}(K,H)} A^{\mathrm{Im}(\beta)}\big) & \longrightarrow & \coprod_{\alpha \in \mathrm{Mono}(K,G)} (G \times_H A)^{\mathrm{Im}(\alpha)} \\ [g, (a, \beta)] & \longmapsto & ([g, a], c_g \circ \iota \circ \beta) \ , \end{array}$$

induces a homeomorphism of homotopy orbits

$$\begin{aligned} EG \times_H \big(\coprod_{\beta \in \mathrm{Mono}(K,H)} A^{\mathrm{Im}(\beta)}\big) &= EG \times_G \big(G \times_H \big(\coprod_{\beta \in \mathrm{Mono}(K,H)} A^{\mathrm{Im}(\beta)}\big)\big) \\ &\cong EG \times_G \big(\coprod_{\alpha \in \mathrm{Mono}(K,G)} (G \times_H A)^{\mathrm{Im}(\alpha)}\big) = (G\backslash\!\backslash(G \times_H A))(K). \end{aligned}$$

We omit the verification that for varying K, the composite maps $(H\backslash\!\backslash A)(K) \longrightarrow (G\backslash\!\backslash(G \times_H A))(K)$ form a morphism of orbispaces, and are thus an equivalence of orbispaces.

Definition 1.10. A orbispace is *compact* if it belongs to the smallest subcategory of the homotopy category of orbispaces that contains the empty orbispace, the orbispace $B_{\mathrm{gl}}G$ for all finite groups G, and that is closed under homotopy pushouts.

Depending on the reader's background and preferences, the term 'homotopy pushout' can either be interpreted in concrete terms via a double mapping cylinder, or in abstract terms as a pushout in the ∞-category of orbispaces.

Example 1.11. Let G be a discrete group and let us denote by $G\mathbf{T}$ the category of G-spaces. The functor

$$G\backslash\!\backslash - \ : \ G\mathbf{T} \ \longrightarrow \ \mathit{orbspc}$$

preserves homotopy pushouts, and the orbispace $G\backslash\!\backslash(G/H)$ is equivalent to $B_{\mathrm{gl}}H$. So if A is a finite proper G-CW-complex, then the global quotient $G\backslash\!\backslash A$ is a compact orbispace. In particular, if G admits a finite G-CW-model for $\underline{E}G$, then the global classifying space $B_{\mathrm{gl}}G$ is a compact orbispace.

In part (iii) of the following theorem and in the rest of the paper, we let $K(n)$ denote the n-th Morava K-theory spectrum at the prime p, with the coefficient ring $K(n)^* = \mathbb{F}_p[v_n^{\pm 1}]$, where v_n is of degree $-2(p^n-1)$. And in part (iv) and below, we let E denote the Morava E-theory spectrum, for an implicit height n and prime p, see [GH04, Section 7]. This is a Landweber exact theory with the coefficient ring $E^* = W(\mathbb{F}_{p^n})[\![u_1,\dots,u_{n-1}]\!][u^{\pm 1}]$, where u is of degree -2.

Theorem 1.12. *Let X be a compact orbispace.*

(i) *There is a number $k \geq 1$ such that for all finite groups H of order larger than k, the space $X(H)$ is empty.*
(ii) *For every finite group K, the total rational homology of the space $X(H)$ is finite-dimensional.*
(iii) *For every prime p and $n \geq 1$, the Morava K-theory $K(n)^*(X(1))$ of the underlying space $X(1)$ is finite-dimensional over $K(n)^*$.*
(iv) *For every prime p and $n \geq 1$, the Morava E-cohomology $E^*(X(1))$ is finitely generated as a graded E^*-module.*

Proof. The overall argument is the same for all four parts: we show that the class of compact orbispaces that has the respective property contains the empty orbispace, it contains the orbispace $B_{\mathrm{gl}}G$ for all finite groups G, and it is closed under homotopy pushouts. The pushout property for (i) follows from the fact that colimits in orbispaces are computed objectwise. For (ii), (iii) and (iv), it follows from the Mayer–Vietoris sequences; in the case of (iv) this exploits that the coefficient ring E^* is Noetherian. And the empty orbispace clearly has all four properties.

It remains to show that all these classes contain $B_{\mathrm{gl}}G$ for any finite G. If G is a finite group of order d, then there are no monomorphisms from groups H to G if the order of H is larger than d. So for such groups H, the space $(B_{\mathrm{gl}}G)(H)$ is empty by (1.3). Independent of the order of H, the space $(B_{\mathrm{gl}}G)(H)$ is always a disjoint union of classifying spaces of finite groups, also by (1.3). So its rational homology is concentrated in dimension 0, where it is finite dimensional. Finally, the underlying space of $B_{\mathrm{gl}}G$ is a classifying space for the finite group G. So its Morava K-theory is finite dimensional over $K(n)^*$ by Ravenel's theorem [Rav82], and its Morava E-cohomology is finitely generated over E^* by [GS99, Corollary 4.4]. □

In the following construction we exploit that the equivalence classes of compact orbispaces form a set.

Definition 1.13. The *orbispace Burnside ring* A_{orb} is the quotient of the free abelian group on the equivalence classes of compact orbispaces by the subgroup generated by the relation $[\emptyset] = 0$, and the relations

$$[W] \ = \ [X] \ + \ [Y] \ - \ [Z] \tag{1.14}$$

for all compact orbispaces X, Y, Z and W such that W can be written as a homotopy pushout of X and Y along Z.

A function f that assigns to every compact orbispace X an element $f(X)$ of an abelian group B is an *additive invariant* if it satisfies the following properties:

(a) If X is equivalent to Y, then $f(X) = f(Y)$.
(b) The empty orbispace is assigned the zero element of B.
(c) If W is the homotopy pushout of two compact orbispaces X and Y along a compact orbispace Z, then $f(W) = f(X) + f(Y) - f(Z)$.

By design, for every such additive invariant, there is a unique group homomorphism $\phi : A_{\mathrm{orb}} \longrightarrow B$ such that $f(X) = \phi[X]$ for every compact orbispace X. In this sense, the orbispace Burnside ring is the universal recipient for additive invariants of compact orbispaces.

Remark 1.15. For the special case $Z = \emptyset$ the pushout relation (1.14) becomes the relation $[X] + [Y] = [X \amalg Y]$, i.e., the coproduct of compact orbispaces represents the sum in the group structure of A_{orb}. We write $X^\diamond$ for the unreduced suspension of an orbispace X, i.e., the (homotopy) pushout of the diagram:

$$\{0,1\} \xleftarrow{\text{proj}} X \times \{0,1\} \xrightarrow{\text{incl}} X \times [0,1]$$

Because $X \times [0,1]$ is equivalent to X, relation (1.14) yields $[X^\diamond] = 2\cdot[*] - [X]$. So the inverse of $[X]$ in A_{orb} is represented by $(X \amalg \{0,1\})^\diamond$, the unreduced suspension of X with two disjoint points added. Because A_{orb} is generated by the classes of compact orbispaces, and because sums and inverses in A_{orb} can be represented by geometric constructions in the realm of compact orbispaces, we conclude that every element of A_{orb} is of the form $[X]$ for some compact orbispace X.

The following theorem determines the structure of the orbispace Burnside ring, and it gives a 'pointwise' criterion for when two compact orbispaces represent the same class in A_{orb}. It is straightforward from the definition that A_{orb} is generated by the classes of the global classifying spaces of all finite groups. The less obvious fact is that the pushout relation (1.14) does not introduce any hidden relations between these generators.

Part (ii) of the next theorem exploits that for a compact orbispace X and a finite group K, the rational homology of the space $X(K)$ has finite total dimension, compare Theorem 1.12 (ii); hence the rational Euler characteristic of $X(K)$ is well-defined.

Theorem 1.16.

(i) *The abelian group A_{orb} is free, and a basis is given by the classes $[B_{\mathrm{gl}}G]$ as G runs over a set of representatives of the isomorphism classes of finite groups.*

(ii) *If X and Y are two compact orbispaces, then $[X] = [Y]$ in A_{orb} if and only for every finite group K, the spaces $X(K)$ and $Y(K)$ have the same rational Euler characteristic.*

Proof. We let F denote the subgroup of A_{orb} generated by the classes $[B_{\mathrm{gl}}G]$ for all finite groups G. We consider the class of all orbispaces X such that $[X] \in F$. This class contains all global classifying spaces of finite groups by design, and it is closed under pushout by the pushout relation (1.14). So $[X] \in F$ for all compact orbispaces X. This shows that A_{orb} is generated by the classes of a global classifying spaces of finite groups, which establishes half of (i).

We prove the other half of (i) together with (ii). For a finite group K, we consider the function that assigns to a compact orbispace X the rational Euler characteristic $\chi_{\mathbb{Q}}(X(K))$ of the space $X(K)$. Then $\chi_{\mathbb{Q}}(\emptyset) = 0$; and if W is a homotopy pushout of X and Y along Z, then $W(K)$ is a homotopy pushout of $X(K)$ and $Y(K)$ along $Z(K)$, so $\chi_{\mathbb{Q}}(W(K)) = \chi_{\mathbb{Q}}(X(K)) + \chi_{\mathbb{Q}}(Y(K)) - \chi_{\mathbb{Q}}(Z(K))$. So sending X to $\chi_{\mathbb{Q}}(X(K))$ is an additive invariant. Hence there is a unique group homomorphism $\phi_K : A_{\mathrm{orb}} \longrightarrow \mathbb{Z}$ such that $\phi_K[X] = \chi_{\mathbb{Q}}(X(K))$ for every compact orbispace X. In particular, if $[X] = [Y]$ in A_{orb}, then $\chi_{\mathbb{Q}}(X(K)) = \chi_{\mathbb{Q}}(Y(K))$ for every finite group K.

The number of isomorphism classes of finite groups is countable. We choose representatives $\{G_n\}_{n\geq 0}$ in some order of non-decreasing cardinality, i.e., $|G_n| \leq |G_{n+1}|$ for all $n \geq 0$. The homomorphisms ϕ_{G_n} are the components of a homomorphism

$$\phi = (\phi_{G_n})_{n\geq 0} \; : \; A_{\mathrm{orb}} \longrightarrow \prod\nolimits_{\mathbb{N}} \mathbb{Z}$$

to a countable product of copies of $\mathbb{Z}$. For $m < n$ we have $|G_m| \leq |G_n|$, and G_m and G_n are not isomorphic. So there is no monomorphism from G_n to G_m. Hence the space $(B_{\mathrm{gl}}G_m)(G_n)$ is empty by (1.3), and thus $\phi_{G_n}[B_{\mathrm{gl}}G_m] = \chi_{\mathbb{Q}}((B_{\mathrm{gl}}G_m)(G_n)) = 0$ for $m < n$. Also by (1.3), the space $(B_{\mathrm{gl}}G_n)(G_n)$ is a disjoint union, indexed by $\mathrm{Out}(G_n)$, of classifying spaces of the center of G_n. So the rational homology of $(B_{\mathrm{gl}}G_n)(G_n)$ is concentrated in dimension 0, where its dimension is the order of $\mathrm{Out}(G_n)$. Thus $\phi_{G_n}[B_{\mathrm{gl}}G_n] = \chi_{\mathbb{Q}}((B_{\mathrm{gl}}G_n)(G_n)) = |\,\mathrm{Out}(G_n)|$. We conclude that the $\mathbb{N} \times \mathbb{N}$ integer matrix with coefficients $\phi_{G_n}[B_{\mathrm{gl}}G_m]$ is upper triangular with nonzero entries on the diagonal. Hence the classes $[B_{\mathrm{gl}}G_n]$ for $n \in \mathbb{N}$ are linearly independent. Since these classes also generate A_{orb}, they form a $\mathbb{Z}$-basis, and the homomorphism $\phi : A_{\mathrm{orb}} \longrightarrow \prod_{\mathbb{N}} \mathbb{Z}$ is injective. This concludes the proof of (i) and (ii). □

Example 1.17. Let G be a discrete group and A a finite proper G-CW-complex. Then the global quotient orbispace $G\backslash\!\backslash A$ is compact, and its class in the orbispace Burnside ring satisfies

$$[G\backslash\!\backslash A] \;=\; \sum_{n\geq 0}(-1)^n \cdot \sum_{G\sigma}[B_{\mathrm{gl}}(\mathrm{stab}(\sigma))] \;\in\; A_{\mathrm{orb}}. \tag{1.18}$$

Here the inner sum runs over all G-orbits of n-cells of A, and $\mathrm{stab}(\sigma)$ is the stabilizer group of the cell σ. Indeed, both sides are additive for disjoint unions and pushouts in A, so it suffices to check the claim for $A = G/H$ for all finite subgroups H of G. In this special case, the claim holds because $G\backslash\!\backslash(G/H)$ is equivalent to $B_{\mathrm{gl}}H$ as an orbispace, see Example 1.9,

Definition 1.19 (Orbispace Euler characteristic)**.** By Theorem 1.16 there is a unique group homomorphism

$$\chi_{\mathrm{orb}} \;:\; A_{\mathrm{orb}} \;\longrightarrow\; \mathbb{Q},$$

such that $\chi_{\mathrm{orb}}[B_{\mathrm{gl}}G] = 1/|G|$ for every finite group G. For a compact orbispace X, we refer to $\chi_{\mathrm{orb}}[X]$ as the *orbispace Euler characteristic* of X.

Many examples of compact orbispaces are global quotients $G\backslash\!\backslash A$ for discrete groups G and finite proper G-CW-complexes A. We now give formulas for their orbispace Euler characteristics.

Example 1.20.

(i) Let G be a discrete group and A a finite proper G-CW-complex. Applying the homomorphism $\chi_{\mathrm{orb}} : A_{\mathrm{orb}} \longrightarrow \mathbb{Q}$ to the formula (1.18) that expresses the global quotient orbispace $G\backslash\!\backslash A$ in terms of the distinguished basis of A_{orb} yields the equality of rational numbers

$$\chi_{\mathrm{orb}}[G\backslash\!\backslash A] \;=\; \sum_{n\geq 0}(-1)^n \cdot \sum_{G\sigma} \frac{1}{|\,\mathrm{stab}(\sigma)|}.$$

If G happens to act *freely* on A, then this becomes

$$\chi_{\mathrm{orb}}[G\backslash\!\backslash A] \;=\; \sum_{n\geq 0}(-1)^n \cdot |\text{equivariant } n\text{-cells}| \;=\; \chi_{\mathbb{Q}}(G\backslash A),$$

the Euler characteristic of the orbit space.

(ii) We let Γ be a finite index subgroup of a discrete group G, and we let A be a finite proper G-CW-complex. Then

$$\chi_{\mathrm{orb}}[G\backslash\!\backslash A] \;=\; \frac{1}{[G:\Gamma]} \cdot \chi_{\mathrm{orb}}[\Gamma\backslash\!\backslash A].$$

Indeed, both sides are additive for disjoint unions and homotopy pushouts in A, so it suffices to check the claim for $A = G/H$ for all finite subgroups H of G. In that case,

$$\begin{aligned} |\Gamma\backslash G| \ &= \sum_{\Gamma g H\in\Gamma\backslash G/H} |\Gamma\backslash(\Gamma g H)| \\ &= \sum_{\Gamma g H\in\Gamma\backslash G/H} |(\Gamma^g\cap H)\backslash H| \ = \sum_{\Gamma g H\in\Gamma\backslash G/H} |H|/|\Gamma\cap {}^g H|, \end{aligned}$$

and hence

$$\begin{aligned} [G:\Gamma]\cdot\chi_{\mathrm{orb}}[G\backslash\!\!\backslash(G/H)] \ &= \ |\Gamma\backslash G|/|H| \ = \sum_{\Gamma g H\in\Gamma\backslash G/H} \frac{1}{|\Gamma\cap {}^g H|} \\ &= \sum_{\Gamma g H\in\Gamma\backslash G/H} \chi_{\mathrm{orb}}[\Gamma\backslash\!\!\backslash(\Gamma/\Gamma\cap {}^g H)] \ = \ \chi_{\mathrm{orb}}[\Gamma\backslash\!\!\backslash(G/H)]. \end{aligned}$$

(iii) We let G be a discrete group that is virtually torsion-free, and we let A be a finite proper G-CW-complex. We let Γ be any torsion-free subgroup of G of finite index; then A is a finite free Γ-CW-complex, so items (i) and (ii) yield the relation

$$\chi_{\mathrm{orb}}[G\backslash\!\!\backslash A] \ = \ \frac{1}{[G:\Gamma]}\cdot\chi_{\mathrm{orb}}[\Gamma\backslash\!\!\backslash A] \ = \ \frac{1}{[G:\Gamma]}\cdot\chi_{\mathbb{Q}}(\Gamma\backslash A).$$

The right hand side is thus independent of the choice of finite index torsion-free subgroup Γ, and usually taken as the definition of the *orbifold Euler characteristic* of the G-space A. We conclude that in this situation our orbispace Euler characteristic specializes to the orbifold Euler characteristic.

(iv) For a *finite* group G and every finite G-CW-complex A, we can take $\Gamma = \{1\}$ in the previous item (iii). We deduce that the orbispace Euler characteristic of the global quotient orbispace is

$$\chi_{\mathrm{orb}}[G\backslash\!\!\backslash A] \ = \ \chi_{\mathbb{Q}}(A)/|G|,$$

where $\chi_{\mathbb{Q}}(A)$ is the ordinary Euler characteristic of the underlying non-equivariant space.

(v) Let G be a countable discrete group that admits a finite G-CW-model for $\underline{E}G$, the universal G-space for proper actions. Suppose that G is also virtually torsion-free, and that Γ is any finite index torsion-free subgroup. Then item (iii) applies to $A = \underline{E}G$, and yields

$$\chi_{\mathrm{orb}}[B_{\mathrm{gl}}G] \ = \ \chi_{\mathrm{orb}}[G\backslash\!\!\backslash\underline{E}G] \ = \ \frac{1}{[G:\Gamma]}\cdot\chi_{\mathbb{Q}}(B\Gamma).$$

Thus the orbispace Euler characteristic of the global classifying space $B_{\mathrm{gl}}G = G\backslash\!\!\backslash\underline{E}G$ is the 'virtual Euler characteristic' of G as defined by Wall [Wal61], also called the orbifold Euler characteristic of G.

Remark 1.21 (Ring structure on A_{orb})**.** The terminology orbispace Burnside *ring* promises a ring structure on the abelian group A_{orb}. We briefly indicate how this ring structures arises; since we do not need the multiplication for this paper, we refrain from giving full details. The topological category $\mathrm{Orb}_{\mathcal{F}in}$ has a continuous symmetric monoidal structure that is given by product of groups on objects; on morphism spaces, the monoidal structure is the effect on homotopy orbits by conjugation of the product of group monomorphisms. The ∞-category obtained as the coherent nerve of $\mathrm{Orb}_{\mathcal{F}in}$ thus inherits a symmetric monoidal structure. We emphasize that this symmetric monoidal structure on $N\,\mathrm{Orb}_{\mathcal{F}in}$ is *not* cartesian; this stems from the fact that the two components of a group monomorphism into a product need not be injective.

Still, a symmetric monoidal structure on an ∞-category induces a symmetric monoidal structure, the *Day convolution product*, on the associated ∞-category of anima-valued presheaves, see e.g. [LNP22, Section 3 and Corollary 3.25]. In our situation, this becomes a symmetric monoidal structure, the *box product* $\boxtimes$, on the ∞-category of orbispaces. The key property of the box product for our purposes are:

- the box product preserves colimits in each variable;
- for finite groups G and K, the box product $(B_{\mathrm{gl}}G) \boxtimes (B_{\mathrm{gl}}K)$ is equivalent to $B_{\mathrm{gl}}(G \times K)$.

From these properties one can reason that the assignment

$$\cdot \ : \ A_{\mathrm{orb}} \times A_{\mathrm{orb}} \ \longrightarrow \ A_{\mathrm{orb}}, \quad [X] \cdot [Y] \ = \ [X \boxtimes Y]$$

is well-defined, biadditive, and that it defines a commutative ring structure on A_{orb} with unit object the terminal orbispace $* \simeq B_{\mathrm{gl}}\{1\}$. In particular, $[B_{\mathrm{gl}}G] \cdot [B_{\mathrm{gl}}K] = [B_{\mathrm{gl}}(G \times K)]$, i.e., the distinguished basis of A_{orb} is a monomial basis for the ring structure.

The orbispace Euler characteristic $\chi_{\mathrm{orb}} : A_{\mathrm{orb}} \longrightarrow \mathbb{Q}$ introduced in Definition 1.19 is a ring homomorphism. It suffices to check the multiplicativity on the generators, where it holds by

$$\begin{aligned} \chi_{\mathrm{orb}}[B_{\mathrm{gl}}G] \cdot \chi_{\mathrm{orb}}[B_{\mathrm{gl}}K] \ &= \ 1/|G| \cdot 1/|K| \ = \ 1/|G \times K| \\ &= \ \chi_{\mathrm{orb}}[B_{\mathrm{gl}}(G \times K)] \ = \ \chi_{\mathrm{orb}}([B_{\mathrm{gl}}G] \cdot [B_{\mathrm{gl}}K]). \end{aligned}$$

2. Orbispaces and Morava K-theory

We let $K(n)$ denote the n-th Morava K-theory spectrum at the prime p. In this section we identify the Morava K-theory Euler characteristic of the underlying space of a compact orbispace Y with the rational Euler characteristic of another space $Y\langle \mathbb{Z}_p^n \rangle$. Here $\mathbb{Z}_p$ is the group of p-adic integers. As we explain in the proof, for $Y = B_{\mathrm{gl}}G$ with G finite, this specializes to [HKR00, Theorem B].

Construction 2.1. For an orbispace Y, we define a space $Y\langle \mathbb{Z}_p^n \rangle$ by

$$Y\langle \mathbb{Z}_p^n \rangle \ = \ \coprod_{N \leq \mathbb{Z}_p^n \text{ f.i.}} Y(\mathbb{Z}_p^n / N),$$

where the disjoint union is indexed by all finite index subgroups of $\mathbb{Z}_p^n$.

Particularly important orbispaces are the global quotients $G \backslash\!\backslash A$, for a discrete group A. The next theorem identifies the spaces $(G \backslash\!\backslash A)\langle \mathbb{Z}_p^n \rangle$ more explicitly. Extending the notation from [HKR00], we let $G_{n,p}$ denote the set of n-tuples $(g_1, \dots, g_n)$ of commuting elements of G, with each g_i having p-power order. The group G acts on $G_{n,p}$ by simultaneous conjugation, and we write $G \backslash G_{n,p}$ for the set of conjugacy classes.

Theorem 2.2. *Let G be a discrete group and A a G-space. Then there is a natural weak equivalence*

$$\coprod_{[g_1, \dots, g_n] \in G \backslash G_{n,p}} EC\langle g_1, \dots, g_n \rangle \times_{C\langle g_1, \dots, g_n \rangle} A^{\langle g_1, \dots, g_n \rangle} \ \xrightarrow{\ \simeq\ } \ (G \backslash\!\backslash A)\langle \mathbb{Z}_p^n \rangle.$$

Here $\langle g_1, \dots, g_n \rangle$ is the subgroup generated by $g_1, \dots, g_n$, and $C\langle g_1, \dots, g_n \rangle$ is its centralizer in G. In particular, there is a weak equivalence

$$\coprod_{[g_1, \dots, g_n] \in G \backslash G_{n,p}} BC\langle g_1, \dots, g_n \rangle \ \xrightarrow{\ \simeq\ } \ (B_{\mathrm{gl}}G)\langle \mathbb{Z}_p^n \rangle.$$

Proof. Every tuple $(g_1,\dots,g_n)$ of commuting p-power order elements determines a continuous homomorphism $\mathbb{Z}_p^n \longrightarrow G$ by sending the standard topological basis $(e_1,\dots,e_n)$ to the tuple $(g_1,\dots,g_n)$. This establishes a bijection

$$\mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^n, G) \cong G_{n,p}, \quad \alpha \mapsto (\alpha(e_1),\dots,\alpha(e_n))$$

that is equivariant for the conjugation action of G on both sides. Also, the image of α is clearly the subgroup generated by $\alpha(e_1),\dots,\alpha(e_n)$. For the course of the proof it will be convenient to work with the G-set $\mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^n, G)$ instead of the G-set $G_{n,p}$.

We let the group G act on the space $\coprod_{\alpha\in\mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^n,G)} A^{\mathrm{Im}\,\alpha}$ by $g\cdot(a,\alpha) = (ga, c_g\circ\alpha)$. We claim that the homotopy orbit space

$$EG \times_G \left(\coprod_{\alpha\in\mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^n,G)} A^{\mathrm{Im}\,\alpha} \right) \tag{2.3}$$

is homeomorphic to $(G\backslash\!\backslash A)\langle\mathbb{Z}_p^n\rangle$. If $\alpha : \mathbb{Z}_p^n \longrightarrow G$ is a continuous homomorphism, then $N = \ker(\alpha)$ is a finite index subgroup, i.e., it is one of the subgroups that index the disjoint union in the definition of $(G\backslash\!\backslash A)\langle\mathbb{Z}_p^n\rangle$. We decompose the disjoint union according to the kernels of the homomorphisms as

$$\coprod_{N\le\mathbb{Z}_p^n \text{ f.i.}} \ \coprod_{\bar\alpha\in\mathrm{Mono}((\mathbb{Z}_p^n)/N,G)} A^{\mathrm{Im}\,\bar\alpha};$$

in the inner disjoint union, $\bar\alpha$ is the unique homomorphism whose composite with the projection $\mathbb{Z}_p^n \longrightarrow \mathbb{Z}_p^n/N$ is α. Clearly, $\bar\alpha$ and α then have the same image. Homotopy orbits commute with disjoint unions. So the space (2.3) is homeomorphic to

$$\begin{aligned}\coprod_{N\le\mathbb{Z}_p^n \text{ f.i.}} EG\times_G &\left(\coprod_{\bar\alpha\in\mathrm{Mono}(\mathbb{Z}_p^n/N,G)} A^{\mathrm{Im}\,\bar\alpha}\right)\\ &= \coprod_{N\le\mathbb{Z}_p^n \text{ f.i.}} (G\backslash\!\backslash A)(\mathbb{Z}_p^n/N) = (G\backslash\!\backslash A)\langle\mathbb{Z}_p^n\rangle.\end{aligned}$$

Now we choose representatives of the G-conjugacy classes of continuous homomorphisms $\alpha : \mathbb{Z}_p^n \longrightarrow G$. These provide a G-equivariant decomposition

$$\coprod_{\alpha\in\mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^n,G)} A^{\mathrm{Im}\,\alpha} = \coprod_{[\alpha]\in G\backslash \mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^n,G)} G\times_{C(\alpha)} A^{\mathrm{Im}\,\alpha}.$$

The homotopy orbit functor $EG\times_G -$ preserves disjoint unions, and the spaces $EG\times_G (G\times_{C(\alpha)} A^{\mathrm{Im}\,\alpha})$ and $EC(\alpha)\times_{C(\alpha)} A^{\mathrm{Im}\,\alpha}$ are weakly equivalent. This proves the theorem. ☐

For every compact orbispace Y, the Morava K-theory $K(n)^*(Y(1))$ of the underlying space $Y(1)$ is finite-dimensional over $K(n)^*$, compare Theorem 1.12 (iii). So we can define the *Morava K-theory Euler characteristic* by

$$\chi_{K(n)}(Y(1)) \ = \ \dim_{K(n)^*}(K(n)^{\mathrm{even}}(Y(1))) \ - \ \dim_{K(n)^*}(K(n)^{\mathrm{odd}}(Y(1))).$$

Theorem 2.4. *Let p be a prime number, and $m,n\ge 1$, and let Y be a compact orbispace.*

(i) *The total rational homology of the space $Y\langle\mathbb{Z}_p^n\rangle$ is finite dimensional, and $K(m)^*(Y\langle\mathbb{Z}_p^n\rangle)$ is finite dimensional over $K(m)^*$.*
(ii) *The following relations between Euler characteristics hold:*

$$\begin{aligned}\chi_{K(n)}(Y(1)) \ &= \ \chi_{\mathbb{Q}}(Y\langle\mathbb{Z}_p^n\rangle);\\ \chi_{K(m+n)}(Y(1)) \ &= \ \chi_{K(m)}(Y\langle\mathbb{Z}_p^n\rangle).\end{aligned}$$

Proof. We start with the special case $Y = B_{\mathrm{gl}}G$ for a finite group G. In this case, Theorem 2.2 identifies the space $(B_{\mathrm{gl}}G)\langle \mathbb{Z}_p^n\rangle$ with a finite disjoint union, indexed by the set $G\backslash G_{n,p}$, of classifying spaces of finite groups; so its rational homology is concentrated in dimension 0, where its dimension equals the cardinality of the set $G\backslash G_{n,p}$. This cardinality agrees with the Morava K-theory Euler characteristic of BG by [HKR00, Theorem B]. Similarly, $K(m)^*((B_{\mathrm{gl}}G)\langle \mathbb{Z}_p^n\rangle)$ is finite dimensional over $K(m)^*$ by Ravenel's theorem [Rav82]. Moreover,

$$\begin{aligned}\chi_{K(m)}((B_{\mathrm{gl}}G)\langle \mathbb{Z}_p^n\rangle) &= \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} \chi_{K(m)}(BC\langle g_1,\dots,g_n\rangle)\\ &= \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} |C\langle g_1,\dots,g_n\rangle\backslash C\langle g_1,\dots,g_n\rangle_{m,p}|\\ &= |G\backslash G_{m+n,p}| = \chi_{K(m+n)}(BG) = \chi_{K(m+n)}((B_{\mathrm{gl}}G)(1)).\end{aligned}$$

The first equality is Theorem 2.2; the second and fourth equalities are [HKR00, Theorem B]; the third equality is the straightforward algebraic fact that upon choosing representatives of the G-conjugacy classes in $G_{n,p}$, the map

$$\coprod_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} C\langle g_1,\dots,g_n\rangle\backslash C\langle g_1,\dots,g_n\rangle_{m,p} \longrightarrow G\backslash G_{m+n,p}$$

sending $[x_1,\dots,x_m]\in C\langle g_1,\dots,g_n\rangle\backslash C\langle g_1,\dots,g_n\rangle_{m,p}$ to $[x_1,\dots,x_m,g_1,\dots,g_n]$ is bijective. So the theorem holds for the global classifying spaces of all finite groups.

The theorem clearly also holds for the empty orbispace. Since the functors sending Y to $Y(1)$ and $Y\langle \mathbb{Z}_p^n\rangle$ preserve colimits, and by the additivity of $\chi_{K(n)}$ and $\chi_{\mathbb{Q}}$ for homotopy pushouts of spaces, the class of compact orbispaces Y for which the theorem holds is closed under homotopy pushouts. So theorem holds for all compact orbispaces. □

Next we examine the special case $Y = G\backslash\!\!\backslash A$, where G is a discrete group and A a finite proper G-CW complex. For a G-space A, we continue to denote by $A^{\langle g_1,\dots,g_n\rangle}$ the fixed space with respect to the subgroup generated by the tuple $(g_1,\dots,g_n)$.

Corollary 2.5. *Let G be a discrete group and let A be a finite proper G-CW complex. Then for all $m,n\geq 1$,*

$$\chi_{K(n)}(EG\times_G A) = \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} \chi_{\mathbb{Q}}(EC\langle g_1,\dots,g_n\rangle \times_{C\langle g_1,\dots,g_n\rangle} A^{\langle g_1,\dots,g_n\rangle}),$$

$$\chi_{K(m+n)}(EG\times_G A) = \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} \chi_{K(m)}(EC\langle g_1,\dots,g_n\rangle \times_{C\langle g_1,\dots,g_n\rangle} A^{\langle g_1,\dots,g_n\rangle}).$$

In particular, if G has a finite model for $\underline{E}G$, then

$$\chi_{K(n)}(BG) \;=\; \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} \chi_{\mathbb{Q}}(BC\langle g_1,\dots,g_n\rangle),$$

$$\chi_{K(m+n)}(BG) = \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} \chi_{K(m)}(BC\langle g_1,\dots,g_n\rangle).$$

Proof. We apply Theorem 2.4 to the compact orbispace $G\backslash\!\!\backslash A$. The underlying space is equivalent to the homotopy orbit space

$$(G\backslash\!\!\backslash A)(1) \;\simeq\; EG\times_G A.$$

Theorem 2.2 provides a splitting

$$(G\backslash\!\!\backslash A)\langle \mathbb{Z}_p^n\rangle \;\simeq\; \coprod_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} EC\langle g_1,\dots,g_n\rangle \times_{C\langle g_1,\dots,g_n\rangle} A^{\langle g_1,\dots,g_n\rangle}.$$

Specializing Theorem 2.4 to the orbispace $G\backslash\!\!\backslash A$ thus yields the result. □

We recall a standard fact:

Lemma 2.6. *Let H be a subgroup of a discrete group G, with centralizer $C(H)$. Let A be a finite proper G-CW complex. Then the $C(H)$-space A^H is a finite proper $C(H)$-CW complex.*

Proof. For every G-CW-complex A, the space A^H is a $C(H)$-CW complex. All points in A have finite stabilizers in G, so all points in A^H in particular have finite stabilizers in $C(H)$. For the finiteness claim about the $C(H)$-CW-structure it suffices to check the case $A = G/K$ for all finite subgroups K of G. In this case, we must show that the set $(G/K)^H$ has finitely many $C(H)$-orbits. If $gK \in G/K$ is H-fixed, then $H^g \leq K$, so conjugation by g is a monomorphism $c_g^{-1} : H \longrightarrow K$, with $c_g^{-1}(h) = g^{-1}hg$. The map

$$C(H)\backslash(G/K)^H \longrightarrow K\backslash \mathrm{Mono}(H,K), \quad C(H)\cdot(gK) \mapsto K \cdot c_g^{-1}$$

is injective, where K acts on the set of monomorphisms by conjugation. Since the target is finite, so is the source. □

Remark 2.7. We let G be a discrete group and A a finite proper G-CW-complex. Then the projection

$$EG \times_G A \longrightarrow G\backslash A$$

induces an isomorphism in rational homology. In particular, if G has a finite model for $\underline{E}G$, then the rational homology of BG is isomorphic to the rational homology of the quotient $\underline{B}G = G\backslash\underline{E}G$ and thus $\chi_{\mathbb{Q}}(BG) = \chi_{\mathbb{Q}}(\underline{B}G)$. More generally, for all $(g_1,\dots,g_n) \in G_{n,p}$, applying Lemma 2.6 to the finite subgroup $H = \langle g_1,\dots,g_n\rangle$ generated by $g_1,\dots,g_n$, shows that the $C\langle g_1,\dots,g_n\rangle$-space $A^{\langle g_1,\dots,g_n\rangle}$ is a finite proper $C\langle g_1,\dots,g_n\rangle$-CW complex. So the projection

$$EC\langle g_1,\dots,g_n\rangle \times_{C\langle g_1,\dots,g_n\rangle} A^{\langle g_1,\dots,g_n\rangle} \longrightarrow C\langle g_1,\dots,g_n\rangle\backslash A^{\langle g_1,\dots,g_n\rangle}$$

induces an isomorphism in rational homology. In the first equation of Corollary 2.5, the rational Euler characteristic of the homotopy orbit space can thus be replaced by the rational Euler characteristic of the strict orbit space $C\langle g_1,\dots,g_n\rangle\backslash A^{\langle g_1,\dots,g_n\rangle}$. In particular if G has a finite model for $\underline{E}G$, then $(\underline{E}G)^{\langle g_1,\dots,g_n\rangle}$ is a finite model for $\underline{E}C\langle g_1,\dots,g_n\rangle$ and $\chi_{\mathbb{Q}}(BC\langle g_1,\dots,g_n\rangle) = \chi_{\mathbb{Q}}(\underline{B}C\langle g_1,\dots,g_n\rangle)$.

If the group G is finite, then so are the groups $C\langle g_1,\dots,g_n\rangle$, and thus the rational Euler characteristic of $BC\langle g_1,\dots,g_n\rangle$ is 1. So in this case, Corollary 2.5 specializes to [HKR00, Theorem B (Part 1)], saying that $\chi_{K(n)}(BG)$ is the number of G-orbits in $G_{n,p}$. Since our proof relies on the results of [HKR00], we are not reproving the theorem of Hopkins, Kuhn and Ravenel, though.

Corollary 2.5 is also closely related to [Ade92, Theorem 4.2] and [Lüc07, Theorem 0.1]. If we take the case $n = 1$, then $K(1)$ is a summand of the mod p complex K-theory. Our result then looks as follows:

$$\chi_{K(1)}(EG \times_G A) = \sum_{[g]\in G\backslash G_{1,p}} \chi_{\mathbb{Q}}(C\langle g\rangle\backslash A^{\langle g\rangle})$$

This agrees with the formula in [Ade92, Theorem 4.2] and exactly picks out the 'p-primary' part of [Lüc07, Theorem 0.1].

Remark 2.8. One can prove Corollary 2.5 without reference to orbispaces, by directly working with finite proper G-CW complexes, as follows. Given G a discrete group with finite $\underline{E}G$, the functors $K(n)^*(EG \times_G A)$ and

$$\bigoplus_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} H^*(C\langle g_1,\dots,g_n\rangle\backslash A^{\langle g_1,\dots,g_n\rangle};\mathbb{Q})$$

are proper equivariant cohomology theories in A; so by additivity and the Mayer–Vietoris property, it suffices to check the desired identity in the case $A = G/H$ for finite subgroups H of G. In that case, the map

$$\coprod_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} C\langle g_1,\dots,g_n\rangle\backslash (G/H)^{\langle g_1,\dots,g_n\rangle} \longrightarrow H\backslash H_{n,p}$$

sending $[xH] \in C\langle g_1,\dots,g_n\rangle\backslash (G/H)^{\langle g_1,\dots,g_n\rangle}$ to $[x^{-1}g_1x,\dots,x^{-1}g_nx]$ is bijective, where we have implicitly chosen representatives of the G-orbits of $G_{n,p}$. Hence

$$\begin{aligned}\chi_{K(n)}(EG\times_G G/H) &= \chi_{K(n)}(BH) = |H\backslash H_{n,p}| \\ &= \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} |C\langle g_1,\dots,g_n\rangle\backslash (G/H)^{\langle g_1,\dots,g_n\rangle}|.\end{aligned}$$

The second equality is [HKR00, Theorem B (Part 1)]. By the proof of Lemma 2.6, $(G/H)^{\langle g_1,\dots,g_n\rangle}$ has finitely many $C\langle g_1,\dots,g_n\rangle$-orbits, and hence

$$|C\langle g_1,\dots,g_n\rangle\backslash (G/H)^{\langle g_1,\dots,g_n\rangle}| = \chi_{\mathbb{Q}}(C\langle g_1,\dots,g_n\rangle\backslash (G/H)^{\langle g_1,\dots,g_n\rangle}).$$

The following proposition records a standard fact for Euler characteristics and follows similarly as the results of [Lüc02b, Section 6.6] and [Bro74]. Indeed, both sides of the formula in (i) are additive invariants in the finite proper G-CW complex A, and for $A = G/H$, where H is finite the formula holds by [HKR00, Theorem B (Part 1)]. So (i) holds, and (ii) is a special case of (i).

Proposition 2.9. *Let G be a discrete group.*

(i) *For every finite proper G-CW complex A and $n \geq 0$,*

$$\chi_{K(n)}(EG\times_G A) = \sum_{G\sigma} (-1)^{n_\sigma}|H^\sigma\backslash H^\sigma_{n,p}|,$$

where the sum runs over all G-orbits of cells σ of A, the number n_σ is the dimension of σ and H^σ is the stabilizer of σ.

(ii) *If G admits a finite model for $\underline{E}G$, then*

$$\chi_{K(n)}(BG) = \sum_{G\sigma} (-1)^{n_\sigma}|H^\sigma\backslash H^\sigma_{n,p}|,$$

where the sum runs over all G-orbits of cells σ of the finite G-CW model for $\underline{E}G$, the number n_σ is the dimension of σ and H^σ is the stabilizer of σ.

Example 2.10. If $G = H *_K L$ is an amalgamated product of finite groups, then Bass–Serre theory [Ser03] provides a one-dimensional finite model for $\underline{E}G$, namely a tree with two equivariant 0-cells with stabilizers H and L and one equivariant 1-cell with stabilizer K, see e.g., [Lüc05b, Example 4.10]. Proposition 2.9 (ii) thus yields

$$\chi_{K(n)}(BG) = |H\backslash H_{n,p}| + |L\backslash L_{n,p}| - |K\backslash K_{n,p}|.$$

Hence by Corollary 2.5, we get a formula

$$|H\backslash H_{n,p}| + |L\backslash L_{n,p}| - |K\backslash K_{n,p}| = \sum_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} \chi_{\mathbb{Q}}(BC\langle g_1,\dots,g_n\rangle). \tag{2.11}$$

Already for $n = 1$ this equation has non-trivial group theoretic meaning, see also [Ade92, Example 4.1]. The relation between $G\backslash G_{1,p}$ on the one hand and $H\backslash H_{1,p}$, $L\backslash L_{1,p}$ and $K\backslash K_{1,p}$ on the other hand is subtle. For example, non-conjugate elements of K might become conjugate in G. This makes centralizers of elements in G very subtle in general, see e.g., [MKS04, Theorem 4.5] and [KS71, Theorem 1].

We write $D_8 = C_4 \rtimes C_2$ for the dihedral group of order 8. We examine the formula (2.11) more closely for the amalgamated product

$$G = D_8 *_{C_4} D_8$$

at the prime 2 and height $n = 1$. The groups D_8 and C_4 have five and four conjugacy classes of elements, respectively. So the left hand side of (2.11) expands to

$$2|D_8\backslash(D_8)_{1,2}| - |C_4\backslash(C_4)_{1,2}| = 2 \cdot 5 - 4 = 6.$$

In any amalgamated product of finite groups $G = H *_K L$, finite order elements are conjugate to elements in H or L, compare the structure theorem for amalgamated products [Ser03, Theorem 2], see also [MKS04, Theorem 4.6] and [GLO15, Lemma 4.3 and Lemma 4.4]. So in our case $G = D_8 *_{C_4} D_8$, every finite order element has 2-power order. The structure theorem also implies that the set $G\backslash G_{1,2}$ has cardinality $2+2+3 = 7$, where 2 is the number of conjugacy classes in D_8 which do not belong to C_4, and 3 is the number of conjugacy classes of C_4 inside D_8. The difference between $|G\backslash G_{1,2}| = 7$ and the left hand side of (2.11) must be accounted for by centralizers. And indeed, the centralizer of a generator of C_4 in G is isomorphic to $C_4 \times \mathbb{Z}$, the free summand being generated by the product of two reflections on different sides of the amalgamated product; and all the other centralizers are rationally acyclic. Because $\chi_{\mathbb{Q}}(B(C_4 \times \mathbb{Z})) = \chi_{\mathbb{Q}}(S^1) = 0$, we deduce that

$$\sum_{[g]\in G\backslash G_{1,2}} \chi_{\mathbb{Q}}(BC\langle g\rangle) = 6 + \chi_{\mathbb{Q}}(B(C_4 \times \mathbb{Z})) = 6,$$

which agrees with the left hand side of (2.11).

3. Hopkins–Kuhn–Ravenel character theory for orbispaces

The purpose of this section is to establish a generalization of the character isomorphism of Hopkins, Kuhn and Ravenel [HKR00, Theorem C] from finite groups to infinite discrete groups. Even more generally, we will prove in Theorem 3.4 that a specific character homomorphism (3.2) for compact orbispaces is an isomorphism. We then specialize it in Corollary 3.5 to global quotients $G\backslash\!\backslash A$ for finite proper G-CW complexes A, and to $B_{\mathrm{gl}}G$ for groups G with finite $\underline{E}G$.

Construction 3.1 (Hopkins–Kuhn–Ravenel character map)**.** We let p be a prime number, and $n \geq 1$. We let E denote the n-th Morava E-theory spectrum, sometimes also called the Lubin–Tate spectrum, at the prime p for height n. Hopkins, Kuhn and Ravenel [HKR00, Section 1.3] define a graded ring $L(E^*)$ as follows. The inverse system of group epimorphisms

$$\ldots \longrightarrow (\mathbb{Z}/p^{k+1})^n \longrightarrow (\mathbb{Z}/p^k)^n \longrightarrow \ldots \longrightarrow (\mathbb{Z}/p)^n \longrightarrow \{1\}$$

induces maps of classifying spaces. Applying E^*-cohomology yields a direct system of morphisms of graded-commutative E^*-algebras

$$E^* \longrightarrow E^*(B(\mathbb{Z}/p)^n) \longrightarrow \ldots \longrightarrow E^*(B(\mathbb{Z}/p^k)^n) \longrightarrow E^*(B(\mathbb{Z}/p^{k+1})^n) \longrightarrow \ldots$$

The colimit of this system is denoted $E^*_{\mathrm{cont}}(B\mathbb{Z}_p^n)$. Then S denotes the subset of $E^2_{\mathrm{cont}}(B\mathbb{Z}_p^n)$ consisting of the first Chern classes of all 1-dimensional complex representations of the groups $(\mathbb{Z}/p^k)^n$, or, equivalently, of all continuous homomorphisms $\alpha : \mathbb{Z}_p^n \longrightarrow U(1)$. The ring $L(E^*)$ is then defined as the localization

$$L(E^*) \;=\; S^{-1}E^*_{\mathrm{cont}}(B\mathbb{Z}_p^n).$$

When the orbispace X is compact, we shall now define a natural homomorphism of graded $L(E^*)$-modules

$$\chi^X_{n,p} \;:\; L(E^*) \otimes_{E^*} E^*(X(1)) \;\longrightarrow\; H^*(X\langle\mathbb{Z}_p^n\rangle; L(E^*)). \tag{3.2}$$

Our definition specializes to the Hopkins–Kuhn–Ravenel character map $\chi^G_{n,p}$ from [HKR00, Theorem C] if we take $X = G\backslash\!\backslash A$ for a finite group G and a finite G-CW-complex A, see Example 3.3.

Because X is compact, Theorem 1.12 (i) provides an number k such that the values $X(K)$ are empty for all finite groups K of order larger than p^k. In particular, $X(\mathbb{Z}_p^n/N)$ is empty whenever N is a subgroup of $\mathbb{Z}_p^n$ of index larger than p^k. If the index of N is less than or equal to p^k, then in particular, $(p^k\mathbb{Z}_p)^n \leq N$. Hence $X\langle\mathbb{Z}_p^n\rangle$ can be rewritten as

$$X\langle\mathbb{Z}_p^n\rangle \;\cong\; \coprod_{M\leq(\mathbb{Z}/p^k)^n} X((\mathbb{Z}/p^k)^n/M).$$

For every normal subgroup M of a finite group G, the functoriality of the orbispace X provides a continuous map

$$BG\times X(G/M) \xrightarrow{B\pi\times\mathrm{Id}} B(G/M)\times X(G/M) = \mathrm{Orb}(e, G/M)\times X(G/M) \;\longrightarrow\; X(1).$$

For varying subgroups of $G = (\mathbb{Z}/p^k)^n$, we obtain a map

$$\coprod_{M\leq(\mathbb{Z}/p^k)^n} B(\mathbb{Z}/p^k)^n \times X((\mathbb{Z}/p^k)^n/M) \;\longrightarrow\; X(1).$$

Using functoriality for this action map and the Künneth formula for E^*-cohomology [HKR00, Corollary 5.11] we obtain a morphism of E^*-algebras

$$\begin{aligned} E^*(X(1)) \;&\longrightarrow E^*(\coprod_{M\leq(\mathbb{Z}/p^k)^n} B(\mathbb{Z}/p^k)^n \times X((\mathbb{Z}/p^k)^n/M)) \\ &\cong\;\; E^*(B(\mathbb{Z}/p^k)^n)\otimes_{E^*} E^*(\coprod_{M\leq(\mathbb{Z}/p^k)^n} X((\mathbb{Z}/p^k)^n/M)) \\ &\xrightarrow{\varphi_k\otimes\mathrm{Id}} L(E^*)\otimes_{E^*} E^*(X\langle\mathbb{Z}_p^n\rangle) \xrightarrow{\text{Chern character}} H^*(X\langle\mathbb{Z}_p^n\rangle; L(E^*)). \end{aligned}$$

Here φ_k is the composite homomorphism of graded-commutative E^*-algebras

$$E^*(B(\mathbb{Z}/p^k)^n) \;\longrightarrow\; E^*_{\mathrm{cont}}(B\mathbb{Z}_p^n) \;\longrightarrow\; S^{-1}E^*_{\mathrm{cont}}(B\mathbb{Z}_p^n) \;=\; L(E^*).$$

The Chern character is defined because $L(E^*)$ is a $(p^{-1}E^*)$-algebra which is rational [HKR00, Theorem C]. The target of the previous composite is an $L(E^*)$-algebra, so scalar extension from E^* to $L(E^*)$ yields the character map (3.2). We omit the verification that the character map just defined remains unchanged if we increase the number k, so that it is independent of the choice of k.

Example 3.3. We explain how the Hopkins–Kuhn–Ravenel character map [HKR00, Theorem C] arises as a special case of the character map (3.2). We let G be a finite group, and we let A be a finite G-CW-complex. The underlying space $X(1) = (G\backslash\!\backslash A)(1)$ is the homotopy orbit space $EG\times_G A$; so the left hand side of our character map coincides with the source of the Hopkins–Kuhn–Ravenel character map. By Theorem 2.2, we have

$$\begin{aligned} (G\backslash\!\backslash A)\langle\mathbb{Z}_p^n\rangle &\simeq \coprod_{[\alpha]\in G\backslash\,\mathrm{Hom}(\mathbb{Z}_p^n,G)} EC(\alpha)\times_{C(\alpha)} A^{\mathrm{Im}(\alpha)} \\ &\simeq EG\times_G (\coprod_{\alpha\in\mathrm{Hom}(\mathbb{Z}_p^n,G)} A^{\mathrm{Im}(\alpha)}). \end{aligned}$$

Because the ring $L(E^*)$ is a $\mathbb{Q}$-algebra and the group G is finite, the functor $H^*(-;L(E^*))$ turns G-homotopy orbits into fixed points and we get

$$\begin{aligned} H^*((G\backslash\!\backslash A)\langle\mathbb{Z}_p^n\rangle;L(E^*)) \ &\cong\ H^*(\coprod_{\alpha\in\mathrm{Hom}(\mathbb{Z}_p^n,G)} A^{\mathrm{Im}(\alpha)};L(E^*))^G \\ &\cong\ L(E^*)\otimes_{E^*}E^*(\coprod_{\alpha\in\mathrm{Hom}(\mathbb{Z}_p^n,G)} A^{\mathrm{Im}(\alpha)})^G \\ &=\ L(E^*)\otimes_{E^*}E^*\left(\mathrm{Fix}_{n,p}(G,A)\right)^G \ =\ \mathrm{Cl}_{n,p}(G,A;L(E^*)). \end{aligned}$$

The second isomorphism uses that each summand $A^{\mathrm{Im}(\alpha)}$ is a finite CW-complex, and that there are only finitely many homomorphisms from $\mathbb{Z}_p^n$ to G. The two equalities are definitions from [HKR00]. We conclude that for $X=G\backslash\!\backslash A$, also the target of our character map (3.2) coincides with the target of the Hopkins–Kuhn–Ravenel character map. Moreover, comparison of the definitions shows that for $X=G\backslash\!\backslash A$, also our map coincides with the one defined by Hopkins, Kuhn and Ravenel.

Theorem 3.4. *Let p be a prime number, and $n\geq 0$. Then for every compact orbispace X, the character map* (3.2) *is an isomorphism.*

Proof. It suffices to show the following statements:

(a) The class of orbispaces for which the character map (3.2) is an isomorphism contains the empty orbispace and is closed under homotopy pushouts.
(b) The class of orbispaces for which the character map (3.2) is an isomorphism contains $B_{\mathrm{gl}}G$ for every finite group G.

Property (a) holds because source and target of the character map are cohomology theories in the orbispace X. Claim (b) is the special case of [HKR00, Theorem C] for $X=*$. □

We let G be a discrete group that admits a finite G-CW-model for $\underline{E}G$. Then the orbispace $X=G\backslash\!\backslash\underline{E}G\simeq B_{\mathrm{gl}}G$ is compact, and so the character map (3.2) is an isomorphism. In this example, the underlying space is a classifying space for the group G, so the source of the character map specializes to $L(E^*)\otimes_{E^*}E^*(BG)$. Theorem 2.2 provides a weak equivalence

$$(B_{\mathrm{gl}}G)\langle\mathbb{Z}_p^n\rangle\ \simeq \coprod_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} BC\langle g_1,\dots,g_n\rangle.$$

Because G admits a finite G-CW-model for $\underline{E}G$, there are only finitely many conjugacy classes of n-tuples of pairwise commuting elements of p-power order. By Theorem 3.4, we obtain:

Corollary 3.5. *Let G be a discrete group that admits a finite G-CW-model for $\underline{E}G$. Then the character map*

$$\chi_{n,p}^{B_{\mathrm{gl}}G}\colon L(E^*)\otimes_{E^*}E^*(BG)\ \xrightarrow{\cong}\ \prod_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} H^*(BC\langle g_1,\dots,g_n\rangle;L(E^*))$$

is an isomorphism.

In Corollary 3.5, the centralizers $C\langle g_1,\dots,g_n\rangle$ need not be finite, and the factors on right hand side need not be free of rank 1 over $L(E^*)$. More generally, for a discrete group G and a finite proper G-CW complex A, applying Theorem 3.4 to the compact orbispace $G\backslash\!\backslash A$ yields a character isomorphism:

$$\chi_{n,p}^{G\backslash\!\backslash A}\colon L(E^*)\otimes_{E^*} E^*(EG\times_G A)\cong \prod_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} H^*(C\langle g_1,\dots,g_n\rangle\backslash A^{\langle g_1,\dots,g_n\rangle};L(E^*))$$

Remark 3.6. Also Corollary 3.5 can be proved without reference to orbispaces. We give a sketch here: For a discrete group G with a finite model for $\underline{E}G$, consider the cohomology theory $L(E^*)\otimes_{E^*} E^*(EG\times_G A)$, where A is a finite proper G-CW complex. We decompose it using [Lüc05a, Theorem 5.5 and Example 5.6]: For a finite G-CW complex A, one has a natural splitting

$$L(E^*)\otimes_{E^*} E^*(EG\times_G A)\cong \prod_{(H)} \mathrm{Hom}_{\mathbb{Q}[W(H)]}(H_*(C(H)\backslash A^H;\mathbb{Q}),T_H^{E^*}), \tag{3.7}$$

where (H) runs over the conjugacy classes of finite subgroups, $C(H)$ is the centralizer of H in G, and $W(H)=N(H)/H\cdot C(H)$, where $N(H)$ is the normalizer. The term $T_H^{E^*}$ is defined as the kernel of the product of restriction maps

$$\mathrm{res}: L(E^*)\otimes_{E^*} E^*(BH)\longrightarrow \prod_{K\leq H,\ K\neq H} L(E^*)\otimes_{E^*} E^*(BK).$$

The right hand side of (3.7) has an appropriate grading which we do not explain here. Using [HKR00, Theorem A and Theorem C], one can see that $T_H^{E^*}$ is trivial unless H is an abelian p-group. In the latter case it is isomorphic as a $\mathbb{Q}[W(H)]$-module to

$$\bigoplus_{\substack{(h_1,\dots,h_n)\in H,\\ \langle h_1,\dots h_n\rangle = H}} L(E^*).$$

Using this, with some more algebraic manipulations one can identify the right hand side of (3.7) with

$$\prod_{[g_1,\dots,g_n]\in G\backslash G_{n,p}} H^*(C\langle g_1,\dots,g_n\rangle\backslash A^{\langle g_1,\dots,g_n\rangle};L(E^*)).$$

We do not go into more technicalities here but invite those readers who prefer this approach to work out the details.

4. Connections to orbifold Euler characteristic

The goal of this section is to relate the orbifold Euler characteristic to the Morava K-theory Euler characteristic. To this end, we introduce the *formal loop space* $\mathcal{L}X$ of an orbispace X in Construction 4.3. We show in Theorem 4.7 that the formal loop space of a compact orbispace is again compact, and the orbispace Euler characteristic of $\mathcal{L}X$ equals the rational Euler characteristic of the underlying space of X. In Theorem 4.9 we apply this general relation to global classifying spaces of discrete groups with finite model for $\underline{E}G$, and deduce a formula for the $K(n)$-Euler characteristic of BG in terms of orbifold Euler characteristics of centralizers of specific $(n+1)$-tuples of commuting finite order elements. In Remark 4.10 we explain how these relationships, for groups which only have p-primary torsion, make χ_{orb} behave like a chromatic height -1 invariant, as opposed to $\chi_{\mathbb{Q}}$ and $\chi_{K(n)}$, which are chromatic height 0 and n invariants, respectively.

Construction 4.1 (Epi-mono factorization)**.** Every group homomorphism factors uniquely as the projection to a quotient group, followed by a monomorphism. This epi-mono factorization extends to the global indexing category, as we recall now. We let N be a normal subgroup of a discrete group K. Then for every group G, precomposition with the projection $\pi_N : K \longrightarrow K/N$ induces an injective map

$\pi_N^* : \mathrm{Mono}(K/N, G) \longrightarrow \mathrm{Hom}(K, G)$. As N varies over all normal subgroups of K, these maps form a bijection

$$\coprod_{N \triangleleft K} \mathrm{Mono}(K/N, G) \xrightarrow{\cong} \mathrm{Hom}(K, G)$$

that is moreover equivariant for the conjugation action of G on both sides. So taking G-homotopy orbits yields a homeomorphism

$$\begin{aligned} \coprod_{N \triangleleft K} \mathrm{Orb}(K/N, G) = \coprod_{N \triangleleft K} EG \times_G \mathrm{Mono}(K/N, G) & \\ \xrightarrow{\cong} EG \times_G \mathrm{Hom}(K, G) = \mathrm{Glo}(K, G). & \qquad (4.2) \end{aligned}$$

By abuse of terminology, we shall also refer to this homeomorphism as the *epi-mono factorization.*

Construction 4.3 (Formal loop space)**.** We define the *formal loop space* $\mathcal{L}Y$ of an orbispace Y. The name is justified by the connection to the formal loop spaces in the sense of Lurie [Lur19, Construction 3.4.3], see Remark 4.5, and because for every finite group G, the underlying space of $\mathcal{L}(B_{\mathrm{gl}}G)$ has the homotopy type of the free loop space of BG, see Theorem 4.6. We alert the reader that $\mathcal{L}Y$ is *different* from the pointwise free loop space. For example, the formal loop space construction preserves colimits (both 1-categorically and ∞-categorically), while the pointwise free loop space does not. In general, the underlying space of $\mathcal{L}Y$ is typically not equivalent to the free loop space of $Y(1)$.

We define the value of the formal loop space at a finite group K by

$$(\mathcal{L}Y)(K) = \coprod_{\substack{N \,\triangleleft\, K \times \mathbb{Z} \text{ f.i.} \\ N \cap (K \times 0) = 1}} Y((K \times \mathbb{Z})/N),$$

The coproduct is indexed by finite index normal subgroups N of $K \times \mathbb{Z}$ that intersect $K \times 0$ in the trivial group.

To define the functoriality of $\mathcal{L}Y$ in the finite group, we fix a finite index normal subgroup M of $G \times \mathbb{Z}$ with $M \cap (G \times 0) = 1$. Product with the group $\mathbb{Z}$ and postcomposition with the projection $\pi_M : G \times \mathbb{Z} \longrightarrow (G \times \mathbb{Z})/M$ pass to continuous maps on morphism spaces. Together with the epi-mono factorization (4.2), this yields a composite:

$$\begin{aligned} & \mathrm{Orb}(K, G) \xrightarrow{-\times\mathbb{Z}} \mathrm{Orb}(K \times \mathbb{Z}, G \times \mathbb{Z}) \\ (4.4) \qquad & \xrightarrow{(\pi_M)_*} \mathrm{Glo}(K \times \mathbb{Z}, (G \times \mathbb{Z})/M) \cong \coprod_{N \triangleleft K \times \mathbb{Z}} \mathrm{Orb}((K \times \mathbb{Z})/N, (G \times \mathbb{Z})/M) \end{aligned}$$

Because $M \cap (G \times 0) = 1$, the image of this composite is contained in the union of those summands indexed by normal subgroups N of $K \times \mathbb{Z}$ that satisfy $N \cap (K \times 0) = 1$. Moreover, because M has finite index in $G \times \mathbb{Z}$, the maps lands in those summands such that N has finite index in $K \times \mathbb{Z}$. We can thus define the partial functoriality on the summand indexed by M as the composite

$$\begin{aligned} & Y((G \times \mathbb{Z})/M) \times \mathrm{Orb}(K, G) \xrightarrow{(4.4)} \\ & \qquad \coprod_{\substack{N \,\triangleleft\, K \times \mathbb{Z} \text{ f.i.} \\ N \cap (K \times 0) = 1}} Y((G \times \mathbb{Z})/M) \times \mathrm{Orb}((K \times \mathbb{Z})/N, (G \times \mathbb{Z})/M) \\ & \qquad \xrightarrow{\circ} \coprod_{\substack{N \,\triangleleft\, K \times \mathbb{Z} \text{ f.i.} \\ N \cap (K \times 0) = 1}} Y((K \times \mathbb{Z})/N) = (\mathcal{L}Y)(K). \end{aligned}$$

As M runs over all those subgroups M of $G \times \mathbb{Z}$ that index the sum defining $(\mathcal{L}Y)(G)$, these maps assemble into the desired functoriality

$$(\mathcal{L}Y)(G) \times \mathrm{Orb}(K, G) \longrightarrow (\mathcal{L}Y)(K).$$

The fact that these maps are also associative boils down to associativity of the epi-mono factorization (4.2).

Remark 4.5. The formal loop orbispace $\mathcal{L}Y$ is an adaptation of the formal loop space construction defined by Lurie in [Lur19, Construction 3.4.3] to our context, the differences being that Lurie works in the ∞-category of global spaces indexed on finite abelian groups, whereas we work in a 1-categorical model for the ∞-category of orbispaces indexed on all finite groups. By definition, the category $\mathrm{Orb}_{\mathcal{F}in}$ is a wide but non-full subcategory of $\mathrm{Glo}_{\mathcal{F}in}$, the full subcategory of Glo spanned by the finite groups, compare Construction 1.1. So restriction of functors from $\mathrm{Glo}_{\mathcal{F}in}$ to $\mathrm{Orb}_{\mathcal{F}in}$ and its left adjoint, continuous left Kan extension, form an adjoint functor pair

$$orbspc = \mathbf{T}_{\mathrm{Orb}} \underset{U}{\overset{(-)_{\mathrm{Glo}}}{\rightleftarrows}} \mathbf{T}_{\mathrm{Glo}} \ .$$

Here $\mathbf{T}_{\mathrm{Glo}}$ denotes the 1-category of global spaces, i.e., continuous functors from $\mathrm{Glo}^{\mathrm{op}}_{\mathcal{F}in}$ to spaces. Since the right adjoint U forgets the functoriality in non-injective group homomorphisms, the left adjoint $(-)_{\mathrm{Glo}}$ can be thought of as 'freely adding inflations', i.e., restriction along surjective group homomorphisms. For more details we recommend Rezk's preprint [Rez14].

For any global space Z and torsion abelian group Λ, Lurie defines the formal loop space $L^{\Lambda}Z$ [Lur19, Construction 3.4.3]. In the special case $\Lambda = \mathbb{Q}/\mathbb{Z}$, the construction can be rewritten as

$$(L^{\mathbb{Q}/\mathbb{Z}}Z)(G) = \mathrm{colim}_{n \in \mathbb{N}_{>0}} Z(G \times \mathbb{Z}/n),$$

where the colimit is over the poset, under divisibility, of positive natural numbers, by inflation along epimorphisms $\mathbb{Z}/kn \longrightarrow \mathbb{Z}/n$. The connection to the formal loop space of Construction 4.3 is that the formal loop space $L^{\mathbb{Q}/\mathbb{Z}}(Y_{\mathrm{Glo}})$ of the globalization of an orbispace Y is naturally equivalent to the globalization of the formal loop orbispace $\mathcal{L}Y$.

Next, we give an explicit formula for the formal loop space of a global quotient orbispace. This is a central technical result which will allow us to compute various Euler characteristics in terms of centralizers. We write G_f for the set of finite order elements in G; the group G acts on G_f by conjugation. We let $C\langle g\rangle$ denote the centralizer of an element $g \in G$. We recall the folklore fact that the free loop space of the classifying space of a discrete group is equivalent to the disjoint union, over conjugacy classes of elements, of the classifying spaces of the centralizers. So the following theorem in particular shows that for finite groups G, the underlying space of $\mathcal{L}(B_{\mathrm{gl}}G)$ has the homotopy type of the free loop space of BG, whence the name.

Theorem 4.6. *Let G be a discrete group and A a G-space. Then there is a natural equivalence of orbispaces*

$$\coprod_{[g] \in G \backslash G_f} C\langle g\rangle \backslash\!\!\backslash A^{\langle g\rangle} \xrightarrow{\ \simeq\ } \mathcal{L}(G \backslash\!\!\backslash A).$$

In particular, there is an equivalence of orbispaces

$$\coprod_{[g] \in G \backslash G_f} B_{\mathrm{gl}}C\langle g\rangle \xrightarrow{\ \simeq\ } \mathcal{L}(B_{\mathrm{gl}}G).$$

Proof. The group G acts on the space $\coprod_{g\in G_f} A^{\langle g\rangle}$ by $\gamma\cdot(g,a) = (\gamma g\gamma^{-1},\gamma a)$. We claim that $\mathcal{L}(G\bbslash A)$ is isomorphic to the global quotient orbispace

$$G\bbslash\Big(\coprod_{g\in G_f} A^{\langle g\rangle}\Big).$$

To prove that, we evaluate at a finite group K. By definition, the value of the above global quotient at K is the homotopy orbit space of the G-space

$$\coprod_{\beta\in\mathrm{Mono}(K,G)}\Big(\coprod_{g\in G_f} A^{\langle g\rangle}\Big)^{\mathrm{Im}(\beta)}.$$

For a given monomorphism $\beta : K \longrightarrow G$, an element (g,a) of the inner disjoint union is fixed by $\mathrm{Im}(\beta)$ if and only if the relation $(\beta(k)g\beta(k)^{-1},\beta(k)a) = (g,a)$ holds for all $k\in K$. This is equivalent to the conditions that $\mathrm{Im}(\beta)$ centralizes g, and that a is also fixed by $\mathrm{Im}(\beta)$. Such pairs combine into homomorphisms

$$K\times\mathbb{Z}\longrightarrow G\ ,\quad (k,m)\mapsto\beta(k)\cdot g^m$$

from the product to G. So the previous G-space equals the G-space

$$\coprod_{\substack{\gamma\,\in\,\mathrm{Hom}^{\mathrm{f.img.}}(K\times\mathbb{Z},G)\\ \gamma|_K\ \text{monic}}} A^{\mathrm{Im}\,\gamma};$$

the disjoint union is taken over homomorphisms $\gamma : K\times\mathbb{Z}\longrightarrow G$ with finite image, and whose restriction to $K\times 0$ is injective. Further, if γ is such a homomorphism, then $N = \ker(\gamma)$ is a finite index normal subgroup whose intersection with $K\times 0$ is trivial, i.e., it is one of the subgroups that index the disjoint union in the definition of $\mathcal{L}(G\bbslash A)(K)$. We decompose the disjoint union according to the kernels of the homomorphism as

$$\coprod_{\substack{N\,\lhd\,K\times\mathbb{Z}\ \text{f.i.}\\ N\cap(K\times 0)=1}}\ \coprod_{\alpha\in\mathrm{Mono}((K\times\mathbb{Z})/N,G)} A^{\mathrm{Im}\,\alpha};$$

in the inner disjoint union, α is the unique homomorphism whose composite with the projection $K\times\mathbb{Z}\longrightarrow(K\times\mathbb{Z})/N$ is γ. Homotopy orbits commute with disjoint unions. So the value of the orbispace $G\bbslash\big(\coprod_{g\in G_f} A^{\langle g\rangle}\big)$ at K is

$$\begin{aligned}\coprod_{\substack{N\,\lhd\,K\times\mathbb{Z}\ \text{f.i.}\\ N\cap(K\times 0)=1}} EG\times_G\Big(\coprod_{\alpha\in\mathrm{Mono}((K\times\mathbb{Z})/N,G)} A^{\mathrm{Im}\,\alpha}\Big)\\ = \coprod_{\substack{N\,\lhd\,K\times\mathbb{Z}\ \text{f.i.}\\ N\cap(K\times 0)=1}} (G\bbslash A)((K\times\mathbb{Z})/N) = \mathcal{L}(G\bbslash A)(K).\end{aligned}$$

We omit the verification that the pointwise homeomorphisms are natural in the group K. Now we choose representatives of the G-conjugacy classes of finite order elements of G. These provide a G-equivariant decomposition

$$\coprod_{g\in G_f} A^{\langle g\rangle} = \coprod_{[g]\in G\backslash G_f} G\times_{C\langle g\rangle} A^{\langle g\rangle}.$$

The global quotient functor $G\bbslash-$ preserves disjoint unions, and the orbispaces $G\bbslash(G\times_{C\langle g\rangle} A^{\langle g\rangle})$ and $C\langle g\rangle\bbslash A^{\langle g\rangle}$ are equivalent, see Example 1.9. This proves the theorem. $\square$

Theorem 4.7. *For every compact orbispace X, the formal loop orbispace $\mathcal{L}X$ is compact, and the relation*

$$\chi_{\mathrm{orb}}[\mathcal{L}X] \ = \ \chi_{\mathbb{Q}}(X(1))$$

between Euler characteristics holds.

Proof. Theorem 4.6 shows that for every finite group G, the orbispace $\mathcal{L}(B_{\mathrm{gl}}G)$ is compact and

$$\chi_{\mathrm{orb}}[\mathcal{L}(B_{\mathrm{gl}}G)] = \sum_{[g]\in G\backslash G_f} \chi_{\mathrm{orb}}[B_{\mathrm{gl}}C\langle g\rangle] = \sum_{[g]\in G\backslash G_f} \frac{1}{|C_G(g)|} = 1 = \chi_{\mathbb{Q}}(BG).$$

So the theorem holds for global classifying spaces of finite groups. The functor $\mathcal{L}$ preserves colimits, so $\mathcal{L}X$ is compact whenever X is. Both sides of the desired equation are additive invariants in X, so the formula holds in general. □

Example 4.8. Theorem 4.7 generalizes a result of Brown [Bro82, Theorem 6.2]. We let G be a discrete group that admits a finite G-CW-model for $\underline{E}G$. Then by Lemma 2.6, for every finite order element $g \in G$, the space $(\underline{E}G)^{\langle g\rangle}$ is a finite $C\langle g\rangle$-CW-model for the universal space for proper actions of the centralizer $C\langle g\rangle$; in particular, $B_{\mathrm{gl}}G = G\backslash\!\backslash\underline{E}G$ and $B_{\mathrm{gl}}C\langle g\rangle$ are compact orbispaces. Using Theorem 4.7 and Theorem 4.6, we conclude that

$$\chi_{\mathbb{Q}}(BG) \ = \ \chi_{\mathrm{orb}}[\mathcal{L}(B_{\mathrm{gl}}G)] \ = \sum_{[g]\in G\backslash G_f} \chi_{\mathrm{orb}}[B_{\mathrm{gl}}C\langle g\rangle].$$

This recovers the above mentioned result of Brown.

In the next lemma and afterwards, we shall use a new piece of notation. Let G be a discrete group, let p be a prime, and let $n \geq 1$. We write $G_{n,p,+1}$ for the set of $(n+1)$-tuples $(x, g_1, \ldots, g_n)$ of commuting elements, with $g_1, \ldots, g_n$ of p-power order, and x a general finite order element. Such tuples are in bijective correspondence with group homomorphism $\mathbb{Z} \times \mathbb{Z}_p^n \longrightarrow G$ with finite image.

Theorem 4.9. *Let G be a discrete group, and let A be a finite proper G-CW complex. Then for every $n \geq 1$,*

$$\chi_{K(n)}(EG \times_G A) = \sum_{[x,g_1,\ldots,g_n]\in G\backslash G_{n,p,+1}} \chi_{\mathrm{orb}}[C\langle x, g_1, \ldots, g_n\rangle\backslash\!\backslash A^{\langle x,g_1,\ldots,g_n\rangle}].$$

In particular, if G has a finite model for $\underline{E}G$, then

$$\chi_{K(n)}(BG) = \sum_{[x,g_1,\ldots,g_n]\in G\backslash G_{n,p,+1}} \chi_{\mathrm{orb}}[B_{\mathrm{gl}}C\langle x, g_1, \ldots, g_n\rangle].$$

Proof. For each tuple $(g_1, \ldots, g_n)$ in $G_{n,p}$, Theorem 4.6 provides a decomposition

$$\begin{aligned}\mathcal{L}(C\langle g_1, \ldots, g_n\rangle\backslash\!\backslash A^{\langle g_1,\ldots,g_n\rangle}) &\simeq \coprod_{[x]\in C\langle g_1,\ldots,g_n\rangle_f^{\mathrm{con}}} C_{C\langle g_1,\ldots,g_n\rangle}\langle x\rangle\backslash\!\backslash(A^{\langle g_1,\ldots,g_n\rangle})^{\langle x\rangle} \\ &= \coprod_{[x]\in C\langle g_1,\ldots,g_n\rangle_f^{\mathrm{con}}} C\langle x, g_1, \ldots, g_n\rangle\backslash\!\backslash A^{\langle x,g_1,\ldots,g_n\rangle},\end{aligned}$$

where $C\langle g_1,\ldots,g_n\rangle_f^{\mathrm{con}}$ denotes the set $C\langle g_1,\ldots,g_n\rangle\backslash C\langle g_1,\ldots,g_n\rangle_f$ of conjugacy classes of finite order elements in $C\langle g_1,\ldots,g_n\rangle$ for brevity. Thus

$$\begin{aligned}
\chi_{K(n)}(EG\times_G A) &= \sum_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} \chi_{\mathbb{Q}}(EC\langle g_1,\ldots,g_n\rangle \times_{C\langle g_1,\ldots,g_n\rangle} A^{\langle g_1,\ldots,g_n\rangle})\\
&= \sum_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} \chi_{\mathrm{orb}}[\mathcal{L}(C\langle g_1,\ldots,g_n\rangle\backslash\!\!\backslash A^{\langle g_1,\ldots,g_n\rangle})]\\
&= \sum_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} \sum_{[x]\in C\langle g_1,\ldots,g_n\rangle_f^{\mathrm{con}}} \chi_{\mathrm{orb}}[C\langle x,g_1,\ldots,g_n\rangle\backslash\!\!\backslash A^{\langle x,g_1,\ldots,g_n\rangle}]\\
&= \sum_{[x,g_1,\ldots,g_n]\in G\backslash G_{n,p,+1}} \chi_{\mathrm{orb}}[C\langle x,g_1,\ldots,g_n\rangle\backslash\!\!\backslash A^{\langle x,g_1,\ldots,g_n\rangle}].
\end{aligned}$$

The first equation is Corollary 2.5; the second equation is Theorem 4.7 for the orbispaces $C\langle g_1,\ldots,g_n\rangle\backslash\!\!\backslash A^{\langle g_1,\ldots,g_n\rangle}$; the final equation follows from the tautological decomposition, namely that the map

$$\coprod_{(g_1,\ldots,g_n)\in G_{n,p}} C\langle g_1,\ldots,g_n\rangle_f \longrightarrow G_{n,p,+1}$$

sending $x\in C\langle g_1,\ldots,g_n\rangle_f$ to $(x,g_1,\ldots,g_n)$ is bijective. Here we recall that $C\langle g_1,\ldots,g_n\rangle_f$ denotes the set of finite order elements in the centralizer of $\langle g_1,\ldots,g_n\rangle$. The conjugation action of G on $G_{n,p,+1}$ permutes the summands on the left according to the conjugation action on $G_{n,p}$, and it acts by conjugation on the centralizers. So upon choosing representatives of the conjugacy classes, the tautological decomposition descends to a bijection

$$\coprod_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} C\langle g_1,\ldots,g_n\rangle_f^{\mathrm{con}} \longrightarrow G\backslash G_{n,p,+1}.$$

□

The previous theorem is a generalization of [HKR00, Lemma 4.13]. Indeed if we take G finite, then using [HKR00, Theorem B], we get

$$|G\backslash G_{n,p}| = \sum_{[x,g_1,\ldots,g_n]\in G\backslash G_{n,p,+1}} \frac{1}{|C\langle x,g_1,\ldots,g_n\rangle|} = \frac{|G_{n,p,+1}|}{|G|}.$$

The last identity is the class equation, and $G_{n,p,+1}$ can be identified with $\mathrm{Hom}(\mathbb{Z}\times\mathbb{Z}_p^n,G)$ which is used in [HKR00, Lemma 4.13].

Remark 4.10. In the case when G has only p-primary torsion and a finite model for $\underline{E}G$, we can now explain why χ_{orb} should be thought as the chromatic height -1 Euler characteristic. In Corollary 2.5 we showed that

$$\chi_{K(n+m)}(BG) = \sum_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} \chi_{K(m)}(BC\langle g_1,\ldots,g_n\rangle)$$

for all $m,n\geq 0$, where $\chi_{K(0)}$ has to be read as the rational Euler characteristic $\chi_{\mathbb{Q}}$. If every torsion element of the group G has p-power order, then $G_{n-1,p,+1}=G_{n,p}$, by definition. So Theorem 4.9 says that

$$\chi_{K(n-1)}(BG) = \sum_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} \chi_{\mathrm{orb}}[B_{\mathrm{gl}}C\langle g_1,\ldots,g_n\rangle].$$

This extends the formula of Corollary 2.5 to the case $m=-1$, as along as we interpret $\chi_{K(-1)}$ as the orbifold Euler characteristic. This explains why at a prime p, we get a chromatic sequence of Euler characteristics starting at -1:

$$\chi_{\mathrm{orb}},\chi_{\mathbb{Q}},\chi_{K(1)},\ldots,\chi_{K(n)},\ldots,$$

where the i-th term can be obtained from the j-th term for $j < i$ using the tuples of $j - i$ many commuting elements of finite order. We will investigate these invariants more closely in Section 5 for $p = 2$ in the case of right angled Coxeter groups. If p is odd, the latter sequence is a special case of the sequence considered in [Yan23]. In this special case, the classifying space BG is a finite colimit of π-finite p-spaces and $\chi_{\mathrm{orb}}[B_{\mathrm{gl}}G] = |BG|$, where $|-|$ is the generalized homotopy cardinality defined by Yanovski.

5. Computations and examples

In this section we apply the theory developed thus far to concrete examples and exhibit explicit formulas for the Morava K-theory Euler characteristics of several classes of infinite discrete groups. In some cases these computations can also be done using the equivariant Atiyah–Hirzebruch spectral sequence by employing nice cellular models for $\underline{E}G$, compare Proposition 2.9. If such models are available, then we make computations in two different ways and compare them. However, in general the combinatorics of cellular models of $\underline{E}G$ is too involved for this direct approach, and then we employ Corollary 2.5 and Corollary 3.5. Examples are arithmetic groups and mapping class groups, see Subsections 5.5–5.8 below. All computations at the height $n > 1$ are new, with the exception of Subsection 5.2, which can be thought of as a consequence of [TY92]. We believe that the calculations for the symplectic and mapping class groups in Subsections 5.7 and 5.8 are new also for the chromatic height $n = 1$.

If G admits a finite model for $\underline{E}G$, we shall write $\chi_{\mathrm{orb}}(BG)$ for the orbispace Euler characteristic $\chi_{\mathrm{orb}}[B_{\mathrm{gl}}G]$ of the global classifying space $B_{\mathrm{gl}}G$. If G is additionally countable and virtually torsion-free, then as observed in Example 1.20 (v), the invariant $\chi_{\mathrm{orb}}(BG)$ coincides with the 'virtual Euler characteristic' of G as defined by Wall [Wal61], also called the orbifold Euler characteristic of G. So our notation is consistent with that of previous literature.

5.1. **Right angled Coxeter groups.** In this subsection we apply the general theory to right angled Coxeter groups. The main reference is [Dav15, Chapter 7]. Let L be a finite graph with vertex set S and set of edges $\mathcal{E}$. The right angled Coxeter group associated to L is the group

$$W(L) = \langle s \in S \mid s^2 = 1 \text{ for all } s \in S, \text{ and } (st)^2 = 1 \text{ for all } \{s,t\} \in \mathcal{E} \rangle.$$

For any subset $T \subset S$, the subgroup W_T generated by T is again a right angled Coxeter group associated to the full subgraph spanned by T. The subset T is called spherical if the subgroup W_T is finite. This is the case if and only if the elements of T commute, i.e., the full subgraph spanned by T is the complete graph. In this case W_T is a product of T many copies of C_2.

We will use a finite model for $\underline{E}W(L)$, known as the Davis complex of $W(L)$ [Dav15, Chapter 7]. We denote it by $\Sigma = \Sigma(W(L))$. We do not recall its construction here but we will now give a description of an equivariant cubical cell structure on it.

In the case when L has only one vertex, we have $W(L) = C_2$. The group C_2 acts on the interval $I = [-1, 1]$ by sign. More generally, given a finite spherical $T \subset S$, the subgroup $W_T = \prod_{|T|} C_2$ acts on $I_T = \prod_{i=1}^{|T|}[-1, 1]$ by the componentwise sign action. The minimal C_2-CW structure on $I = [-1, 1]$ with one fixed 0-cell, one free 0-cell and one free 1-cell induces a W_T-CW structure on I_T.

In general, one has a finite increasing filtration $\bigcup_{m\geq 0}\Sigma^m = \Sigma$, where $\Sigma^{|S|} = \Sigma$ and for all $m \geq 0$, there are cofiber sequences of $W(L)$-spaces

$$\Sigma^{m-1} \hookrightarrow \Sigma^m \to \bigvee_{\text{spherical } T\subset S,\ |T|=m} W(L) \ltimes_{W_T} I_T/\partial I_T, \tag{5.1}$$

where ∂I_T is the boundary of the cube I_T.

The papers [SG07] and [DL17] use this filtration to compute the equivariant K-homology and K-theory of $\underline{E}W(L)$. We will first use the same method to compute $K(n)^*(BW(L))$ and $E^*(BW(L))$, and then double-check the results by using Corollary 2.5 and Corollary 3.5 instead.

We will consider cochain complexes with values in the abelian category of graded $K(n)^*$-modules. Since $W(L)$ has only 2-torsion, we work at the prime 2. The calculations start from the well-known fact that

$$K(n)^*(BC_2) \cong K(n)^*[x]/(x^{2^n}),$$

where x is of degree 2, namely the Euler class of the complex line bundle over BC_2 associated to the complex sign representation.

The next lemma computes Bredon cohomology for cubes and elementary abelian subgroups, corresponding to the spherical subsets.

Lemma 5.2. *Let $T \subset S$ be a spherical subset of cardinality l of a finite graph. Then*

$$\widetilde{H}^i_{W_T}(I_T/\partial I_T, K(n)^*(-)) = \begin{cases} (K(n)^*\{x,\dots,x^{2^n-1}\})^{\otimes l} & \text{for } i = 0, \\ 0 & \text{for } i \neq 0, \end{cases}$$

where the tensor power is formed over $K(n)^*$.

Proof. We start with the case where T has one element, in which case $I_T = [-1,1]$ with sign action by $W_T = C_2$. We use the minimal C_2-CW-structure on $I/\partial I$ with two fixed 0-cells (one of which is the basepoint) and one free 1-cell. The reduced Bredon cochain complex of $I/\partial I$ in the category of $K(n)^*$-modules is concentrated in degrees 0 and 1, where the groups are the kernel and cokernel, respectively of the homomorphism

$$\mathrm{res}\colon K(n)^*\{1,x,\dots,x^{2^n-1}\} = K(n)^*(BC_2) \longrightarrow K(n)^*.$$

The restriction homomorphism is surjective and sends all positive powers of x to zero, which proves the claim in the special case.

In the general case we use the product W_T-cell structure on I_T and the Künneth formula for the Morava K-theories. The stabilizers of I_T are elementary abelian of the form $W_{T'}$ where $T' \subset T$. The W_T-cellular structure of I_T is the product of the C_2-cellular structures of $I = [-1,1]$ and each product cell has an elementary abelian stabilizer. Using the Künneth formula $K(n)^*(BG \times BH) \cong K(n)^*(BG) \otimes_{K(n)^*} K(n)^*(BH)$, we see that the reduced Bredon cochain complex

$$\widetilde{C}^\bullet_{W_T}(I_T/\partial I_T, K(n)^*(-))$$

is isomorphic to the tensor power

$$\widetilde{C}^\bullet_{C_2}(I/\partial I, K(n)^*(-))^{\otimes l}$$

in the category of cochain complexes of graded $K(n)^*$-modules. Using the special case and that $K(n)^*$ is a graded field, the Künneth theorem for cochain complexes proves the desired result. $\square$

Proposition 5.3. *Let L be a finite graph with vertex set S, and $n \geq 0$. Then for the prime $p = 2$, the $K(n)^*$-module $K(n)^*(BW(L))$ is concentrated in even degrees and is of dimension*

$$\dim_{K(n)^*}(K(n)^*(BW(L))) = \sum_{l=0}^{|S|} s(l) \cdot (2^n - 1)^l,$$

where $s(l)$ is the number of spherical subsets $T \subset S$ of size l. Hence the $K(n)$-Euler characteristic is

$$\chi_{K(n)}(BW(L)) = \sum_{l=0}^{|S|} s(l)(2^n - 1)^l.$$

Proof. We will show that the equivariant Atiyah–Hirzebruch spectral sequence based on the Davis complex $\Sigma = \Sigma(W(L))$ that models $\underline{E}W(L)$ collapses. To determine the E^2-term of this spectral sequence we use the filtration Σ^m of Σ. In a first step we show by induction on m that the Bredon cohomology groups $H^i_{W(L)}(\Sigma^m, K(n)^*(-))$ are trivial for $i \neq 0$, and $H^0_{W(L)}(\Sigma^m, K(n)^*(-))$ is even and of dimension

$$\dim_{K(n)^*}(H^0_{W(L)}(\Sigma^m, K(n)^*(-))) = \sum_{l=0}^{m} s(l) \cdot (2^n - 1)^l.$$

The induction starts with $m = -1$, where Σ^{-1} is empty, and there is nothing to show. For the inductive step we use the cofiber sequence (5.1). By induction and Lemma 5.2, the Bredon cohomology groups of Σ^{m-1} and of the cofiber vanish except in cohomological degree $i = 0$, and they are concentrated in even internal degrees. So the Bredon cohomology long exact sequence decomposes into short exact sequence and shows that the same is true for Σ^m. Moreover, the $K(n)^*$-dimension of the 0-th Bredon cohomology of Σ^m is the sum of the dimensions of the 0-th Bredon cohomology of Σ^{m-1} and the cofiber, which we know by induction and Lemma 5.2. This completes the inductive step.

Because $H^*_{W(L)}(\Sigma, K(n)^*(-))$ is concentrated in Bredon cohomology degree 0, the Atiyah–Hirzebruch spectral sequence for $K(n)^*(BW(L))$ collapses at the E^2-term. Since the spectral sequence consists of modules over the graded field $K(n)^*$, the evenness property and dimension of the E^2-term are inherited by the abutment. Hence $K(n)^*(BW(L))$ is concentrated in even degrees, and

$$\begin{aligned}\dim_{K(n)^*}(K(n)^*(BW(L))) &= \dim_{K(n)^*}(H^0_{W(L)}(\Sigma^{|S|}, K(n)^*(-))) \\ &= \sum_{l=0}^{|S|} s(l) \cdot (2^n - 1)^l.\end{aligned}$$

□

As a reality check we will now rediscover the formula for the $K(n)$-Euler characteristic of $BW(L)$ from Proposition 5.3 in a different way by using our theory, specifically Corollary 2.5. For this we need to count the number of $W(L)$-conjugacy classes of n-tuples of 2-power order elements in $W(L)$:

Proposition 5.4. *Let L be a finite graph with vertex set S, and $n \geq 0$. Then*

$$|W(L)\backslash W(L)_{n,2}| = \sum_{l=0}^{|S|} s(l)(2^n - 1)^l.$$

Proof. By [DL17, pages 13-14], every finite subgroup of W is subconjugate to W_T for some spherical subset T of S. Any element $s \in S$ gives $(2^n - 1)$ many elements of $W_{n,2}$ by taking tuples $(w_1, \ldots, w_n)$, where at least one $w_i = s$ and $w_j = 1$ or

s for any j. Now consider any general element $(w_1, \ldots, w_n) \in W_{n,2}$, where all the coordinates belong to some W_T, where T is spherical. Let $\{s_1, \ldots, s_l\}$ be the set of all letters from the reduced expressions of w_i-s. Then $(w_1, \ldots, w_n)$ can be uniquely written up to permutation in the Cartesian power $W^{\times n}$ as a product $v_1 v_2 \cdots v_l$, where $v_i \in \{1, s_i\}^{\times n} - \{(1, \ldots, 1)\}$. This implies that the number of tuples $(w_1, \ldots, w_n) \in W_{n,2}$ where all the coordinates belong to some W_T with T spherical and which involve exactly l many letters in total from S is equal to $(2^n - 1)^l$, meaning that every spherical subset of size l gives exactly $(2^n - 1)^l$ many elements in $W_{n,2}$. None of these tuples are conjugate since the abelianization of W is an $\mathbb{F}_2$-vector space generated by S. In particular, the normalizer of W_T for T spherical, agrees with the centralizer which also follows from [Dav15, Proposition 4.10.2]. This completes the proof. $\square$

Remark 5.5. Proposition 5.4 can be now used to recover the formula

$$\chi_{K(n)}(BW(L)) = \sum_{l=0}^{|S|} s(l)(2^n - 1)^l$$

without using the equivariant cubical cell structure of Σ. Indeed by Corollary 2.5, it suffices to show that $\chi_{\mathbb{Q}}(BC(H)) = 1$ for any $H \leq W_T$, where T is a spherical subset. It follows from [Bar07, Proposition 25 and Theorem 32] and [Dav15, Theorem 4.1.6] that $C(H) = W_{S'}$ for some subset $S' \subset S$. But $\chi_{\mathbb{Q}}(BW_{S'}) = 1$ since right-angled Coxeter groups are rationally acyclic.

Example 5.6. For $n = 0$, the Euler characteristic formula of Proposition 5.3 specializes to $\chi_{\mathbb{Q}}(BW(L)) = 1$, a consequence of the well-known fact that $BW(L)$ is rationally acyclic. If we pretend that we are allowed to let $n = -1$, then we recover the well-known orbifold Euler characteristic

$$\chi_{\mathrm{orb}}(BW(L)) = \sum_{l=0}^{|S|} s(l) \cdot \tfrac{(-1)^l}{2^l},$$

see for example [Dav15, (16.8)]. This is in tune with our slogan that the orbifold Euler characteristic is a 'height -1 invariant', compare Remark 4.10. For more concreteness, we examine a special case when the graph L does not contain triangles. In this case we can easily understand the number $s(l)$ for all $l \geq 0$. Indeed, $s(0) = 1$, $s(1) = |S|$ and $s(2) = |\mathcal{E}|$. For $n \geq 3$, we have $s(n) = 0$. Hence Proposition 5.4 and Remark 5.5 give us

$$\chi_{K(n)}(BW(L)) = |W \backslash W_{n,2}| = 1 + |S|(2^n - 1) + |\mathcal{E}|(2^n - 1)^2.$$

For $n = -1$, one recovers the orbifold Euler characteristic [Dav15, (16.8)]

$$\chi_{\mathrm{orb}}(BW(L)) = 1 - \tfrac{|S|}{2} + \tfrac{|\mathcal{E}|}{4}.$$

Now we apply the character theory of Section 3. Let E denote the Morava E-theory at prime 2 and height n. We write $W = W(L)$ for brevity. Corollary 3.5 provides an isomorphism

$$L(E^*) \otimes_{E^*} E^*(BW) \cong \bigoplus_{[w_1,\ldots,w_n] \in W \backslash W_{n,2}} H^*(BC\langle w_1, \ldots, w_n\rangle; L(E^*)).$$

As we observed in Remark 5.5, the space $BC\langle w_1, \ldots, w_n\rangle$ has trivial rational cohomology. Hence we get an isomorphism

$$L(E^*) \otimes_{E^*} E^*(BW) \cong L(E^*)\{W \backslash W_{n,2}\},$$

where the right hand side is the free $L(E^*)$-module on the set $W \backslash W_{n,2}$, with generators of degree 0. The cardinality of this set, and hence the rank of $L(E^*) \otimes_{E^*} E^*(BW)$ is given by Proposition 5.4.

Because $E^*(BW(L))$ is finitely generated over E^* by Theorem 1.12 (iv), and because $K(n)^*(BW(L))$ is even by Proposition 5.3, we can conclude by [Str98, Proposition 3.5] that $E^*(BW(L))$ is in fact even and free over E^*. Since we also know its rank after scalar extension to the ring $L(E^*)$, we deduce:

Theorem 5.7. *Let L be a finite graph with vertex set S, and $n \geq 0$. The Morava E-cohomology $E^*(BW(L))$ at the prime 2 is even and free of rank*

$$\sum_{l=0}^{|S|} s(l)(2^n - 1)^l.$$

The previous theorem can be directly established without using [Str98, Proposition 3.5]. In fact one can use the cellular decomposition of the Davis complex Σ and the same strategy as in Proposition 5.3. One only needs to use the generalized version of Künneth theorem proved in [HKR00, Corollary 5.11]. See also [KLL21] for a general computation of equivariant (co)homology of graph products.

5.2. **The special linear group $SL_3(\mathbb{Z})$.** In this subsection we recover the well-known calculations of $K(n)^*(BSL_3(\mathbb{Z}))$ and $E^*(BSL_3(\mathbb{Z}))$. These groups have been computed by Tezuka and Yagita in [TY92] using the finite cellular model of $\underline{E}SL_3(\mathbb{Z})$ constructed by Soulé [Sou78]. The height 1 computations can be also recovered by [SG08]. Some of these calculations are also included in [Sch06, Section VII].

The main strategy in [TY92] and [SG08] is to use the equivariant Atiyah–Hirzebruch spectral sequence, compute the Bredon cohomology of $\underline{E}SL_3(\mathbb{Z})$ with various coefficients, and then the generalized cohomology. The spectral sequences are relatively easy to handle since the virtual cohomological dimension of $SL_3(\mathbb{Z})$ is equal to 3 [BS73]. The explicit finite model of Soulé [Sou78] is based on finite subgroup classification of $SL_3(\mathbb{Z})$ by [Tah71]. We do not recall here the details of this model and refer to [SG08, Table 1] and [Sou78, Theorem 2] for the equivariant cellular structure and the list of stabilizers.

We first recall the approach of [TY92] for computing $\chi_{K(n)}(BSL_3(\mathbb{Z}))$. After this we offer an alternative way to arrive at these calculations: instead of completely writing out the equivariant cellular chain complex of $\underline{E}SL_3(\mathbb{Z})$, we compute the centralizers of finite abelian subgroups and use Corollary 2.5 to calculate $\chi_{K(n)}(BSL_3(\mathbb{Z}))$. For $n = 1$ this has been already done in [Ade92, Example 4.3].

The group $SL_3(\mathbb{Z})$ has only 2 and 3-torsion [Tah71]. Hence the p-primary case for $p \geq 5$ is equivalent to the rational case. By [Sou78, p. 8 Corollary], the space $\underline{B}SL_3(\mathbb{Z}) = SL_3(\mathbb{Z})\backslash\underline{E}SL_3(\mathbb{Z})$ is contractible and hence rationally acyclic. We start with the easier case of $p = 3$. The following is proved in [TY92, Section 5]:

Proposition 5.8. *Let $n \geq 1$ and $p = 3$. Then $K(n)^*(BSL_3(\mathbb{Z}))$ is even and $\chi_{K(n)}(BSL_3(\mathbb{Z})) = 3^n$.*

Proof. We recall the proof from [TY92]. Proposition 2.9 yields the relation

$$\chi_{K(n)}(BSL_3(\mathbb{Z})) = \sum_{G\sigma} (-1)^{n_\sigma} |H^\sigma \backslash H^\sigma_{n,3}|,$$

where the sum runs over all G-orbits of cells σ of Soulé's model of $\underline{E}SL_3(\mathbb{Z})$, the number n_σ is the dimension of σ and H^σ is the stabilizer of σ. By looking at [SG08, Table 1], we see that there are only six equivariant cells whose stabilizers have 3-torsion: three 0-cells with stabilizer S_4, one 0-cell with stabilizer D_{12} and two 1-cells with stabilizer S_3. A simple calculation shows that

$$|S_4\backslash(S_4)_{n,3}| = |D_{12}\backslash(D_{12})_{n,3}| = |S_3\backslash(S_3)_{n,3}| = \tfrac{3^n-1}{2} + 1.$$

This follows from the fact that Sylow 3-subgroups of all these three groups are isomorphic to C_3 and the two generators of C_3 are conjugate in S_3 (and hence in S_4 and D_{12}). All the remaining equivariant cells have stabilizers without 3-torsion. Again [SG08, Table 1] tells us how many such equivariant cells we have: one 0-cell, six 1-cells, five 2-cells, and one 3-cell. All in all, we get

$$\begin{aligned}\chi_{K(n)}(BSL_3(\mathbb{Z})) &= \sum_{G\sigma}(-1)^{n_\sigma}|H^\sigma\backslash H^\sigma_{n,3}| \\ &= 4(\tfrac{3^n-1}{2}+1) - 2(\tfrac{3^n-1}{2}+1) + 1 - 6 + 5 - 1 \\ &= 2(\tfrac{3^n-1}{2}+1) - 1 = 3^n - 1 + 2 - 1 = 3^n.\end{aligned}$$

In fact as observed in [TY92, Section 5], the $K(n)^*$-module $K(n)^*(BSL_3(\mathbb{Z}))$ can be described using [Sou78, Lemma 6] and one can see that there is an isomorphism $K(n)^*(BSL_3(\mathbb{Z})) \cong K(n)^*(BS_3) \oplus \widetilde{K(n)}^*(BS_3)$, where $\widetilde{K(n)}^*(-)$ is the reduced cohomology. By the remark immediately after [HKR00, Theorem E], all symmetric groups are 'good' in the sense of [HKR00, Definition 7.1], and hence their Morava K-theory is concentrated in even degrees. In particular, we know that $K(n)^*(BS_3)$ is even. □

We would like to compare the latter result with the formula of Corollary 2.5. By the classification of finite subgroups of [Tah71], there are only two conjugacy classes of non-trivial finite 3-subgroups of $SL_3(\mathbb{Z})$, namely the cyclic groups of order 3 generated by the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

respectively. The following is observed in [Ade92] and [Upa96] without a proof:

Lemma 5.9. *Centralizers of finite* 3*-subgroups of* $SL_3(\mathbb{Z})$ *are finite.*

Proof. We need to show that the centralizers of the matrices A and B are finite. The centralizer of A is isomorphic to the centralizer of

$$A' = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

in $SL_2(\mathbb{Z})$. This matrix has order 3, and explicitly solving the equation $ZA' = A'Z$ reveals that the centralizer of A' is generated by the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

of order 6. In particular, the centralizer of A' in $SL_2(\mathbb{Z})$ is finite.

The second case requires a bit more work. The matrix B has three different eigenvalues $1, \zeta, \zeta^2$, where $\zeta = e^{2\pi i/3}$ is a primitive third root of unity. Then over $\mathbb{C}$, the matrix B is diagonalizable

$$Z^{-1}BZ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix},$$

where Z is the matrix of eigenvectors,

$$Z = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix}.$$

The centralizer inside $SL_3(\mathbb{Q}(\zeta))$ of the diagonal matrix above consists of general diagonal matrices of the form

$$C = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

with $a, b, c \in \mathbb{Q}(\zeta)$ and $abc = 1$. The centralizer of B in $SL_3(\mathbb{Z})$ thus consists of all matrices of the form ZCZ^{-1} which additionally have integral coefficients. This reveals the general form of centralizing matrices of B as

$$\frac{1}{3}\begin{pmatrix} a+b+c & a+b\zeta^2+c\zeta & a+b\zeta+c\zeta^2 \\ a+b\zeta+c\zeta^2 & a+b+c & a+b\zeta^2+c\zeta \\ a+b\zeta^2+c\zeta & a+b\zeta+c\zeta^2 & a+b+c \end{pmatrix},$$

for $a, b, c \in \mathbb{Q}(\zeta)$, subject to the condition that all the matrix entries $\frac{1}{3}(a+b+c)$, $\frac{1}{3}(a+b\zeta+c\zeta^2)$ and $\frac{1}{3}(a+b\zeta^2+c\zeta)$ are integers, and $abc = 1$. Then also

$$a = \frac{1}{3}(a+b+c) + \frac{1}{3}(a+b\zeta+c\zeta^2) + \frac{1}{3}(a+b\zeta^2+c\zeta)$$

lies in $\mathbb{Z}$, and

$$b = \frac{1}{3}(a+b+c) + \frac{\zeta^2}{3}(a+b\zeta+c\zeta^2) + \frac{\zeta}{3}(a+b\zeta^2+c\zeta)$$

lies in $\mathbb{Z}[\zeta]$, and similarly, $c \in \mathbb{Z}[\zeta]$. A special case of Dirichlet's unit theorem says that $\mathbb{Z}[\zeta]^\times = \{\pm 1, \pm\zeta, \pm\zeta^2\}$ and hence there are only finitely many $a, b, c \in \mathbb{Z}[\zeta]$ with abc=1. This shows that $C\langle B\rangle$ is finite. □

Remark 5.10. Using Lemma 5.9 and Corollary 2.5, we recover the Euler characteristic $\chi_{K(n)}(BSL_3(\mathbb{Z}))$. Indeed, Lemma 5.9 tells us that for any $(g_1, \ldots, g_n) \in SL_3(\mathbb{Z})_{n,3}$ with at least one $g_i \neq 1$, the centralizer $C\langle g_1, \ldots, g_n\rangle$ is finite. Moreover, $C(1) = SL_3(\mathbb{Z})$ and as mentioned above by [Sou78, p. 8 Corollary], the space $\underline{B}SL_3(\mathbb{Z})$ is contractible and hence $BSL_3(\mathbb{Z})$ is rationally acyclic. All in all we get

$$\chi_{K(n)}(BSL_3(\mathbb{Z})) = \sum_{[g_1,\ldots,g_n]\in SL_3(\mathbb{Z})\backslash SL_3(\mathbb{Z})_{n,3}} \chi_{\mathbb{Q}}(BC\langle g_1,\ldots,g_n\rangle) = \sum_{[g_1,\ldots,g_n]\in SL_3(\mathbb{Z})\backslash SL_3(\mathbb{Z})_{n,3}} 1 = |SL_3(\mathbb{Z})\backslash SL_3(\mathbb{Z})_{n,3}|.$$

Again using the classification of finite subgroups of $SL_3(\mathbb{Z})$ [Tah71], we see that

$$|SL_3(\mathbb{Z})\backslash SL_3(\mathbb{Z})_{n,3}| = \frac{3^n+1}{2} + \frac{3^n+1}{2} - 1 = 3^n.$$

We note that this approach still uses the cellular structure of [Sou78] since the proof of the contractibility of $\underline{B}SL_3(\mathbb{Z})$ does. However, we do not need to go into the combinatorics of the cell structure to compute $\chi_{K(n)}(BSL_3(\mathbb{Z}))$.

Finally, using Corollary 3.5, Remark 5.10 and Proposition 5.8 as well as [Str98, Proposition 3.5], we obtain:

Proposition 5.11. *Let $n \geq 1$ and $p = 3$. Then $E^*(BSL_3(\mathbb{Z}))$ at the prime 3 is even and free of rank 3^n as a graded E^*-module.*

The latter can be also recovered from [TY92, Section 5] since E is Landweber exact.

Now we switch to the prime $p = 2$ case. This case is more involved, though centralizers can be computed with the same methods as in Lemma 5.9. Again we recall the calculation from [TY92] using the cellular structure of [Sou78] and then compare the result to the formula we get from Corollary 2.5.

Proposition 5.12. *Let $n \geq 1$ and $p = 2$. Then $K(n)^*(BSL_3(\mathbb{Z}))$ is even and $\chi_{K(n)}(BSL_3(\mathbb{Z})) = 2^{2n+1} - 2^{n+1} + 1$.*

Proof. We need formulas from [HKR92, Section 4]: $\chi_{K(n)}(S_3) = 2^n$,

$$\chi_{K(n)}(S_4) = \tfrac{7 \cdot 4^n - 3 \cdot 2^n + 2}{6} \quad \text{and} \quad \chi_{K(n)}(D_8) = \tfrac{3 \cdot 4^n - 2^n}{2}.$$

Along similar lines one shows that $\chi_{K(n)}(D_{12}) = 4^n$. We recall again the equivariant cellular structure on $\underline{E}SL_3(\mathbb{Z})$, compare [Sou78] and [SG08, Table 1]: There are three equivariant 0-cells with stabilizer S_4, one equivariant 0-cell with stabilizer D_8, and another equivariant 0-cell with stabilizer D_{12}. One has two equivariant 1-cells with stabilizer C_2, two with stabilizer D_8, two with stabilizer S_3, and another two with stabilizer $C_2 \times C_2$. Further, we have three equivariant 2-cells with stabilizer C_2, one with stabilizer $C_2 \times C_2$, and one free equivariant 2-cell. Finally, there is only one equivariant 3-cell which is free. All in all by Proposition 2.9, we get

$$\begin{aligned}
\chi_{K(n)}(BSL_3(\mathbb{Z})) =& \sum_{G\sigma} (-1)^{n_\sigma} |H^\sigma \backslash H^\sigma_{n,2}| \\
=& 3 \cdot \tfrac{7 \cdot 4^n - 3 \cdot 2^n + 2}{6} + 4^n + \tfrac{3 \cdot 4^n - 2^n}{2} \\
& - 2 \cdot 4^n - 2 \cdot 2^n - 2 \cdot 2^n - 2 \cdot \tfrac{3 \cdot 4^n - 2^n}{2} \\
& + 3 \cdot 2^n + 1 + 4^n - 1 \\
=& \; 2^{2n+1} - 2^{n+1} + 1.
\end{aligned}$$

Here again the sum runs over all G-orbits of cells σ of Soulé's model of $\underline{E}SL_3(\mathbb{Z})$.

To show that $K(n)^*(BSL_3(\mathbb{Z}))$ is even, Tezuka and Yagita use a 1-dimensional subcomplex of $\underline{E}SL_3(\mathbb{Z})$ whose Bredon cohomology agrees with the Bredon cohomology of $\underline{E}SL_3(\mathbb{Z})$. By [HKR00, Theorem E] we know that S_4 and D_8 are 'good' groups, and hence $K(n)^*(BS_4)$, and $K(n)^*(BD_8)$ are even. The rest follows from [TY92, Theorem 2.6]. □

Now we compare the latter result with Corollary 2.5.

Lemma 5.13. *Centralizers of finite abelian 2-subgroups of $SL_3(\mathbb{Z})$ are rationally acyclic.*

Proof. According to [Upa96] the only non-finite centralizers are those of subgroups of order at most 2. The trivial subgroup case follows by the fact that $\underline{B}SL_3(\mathbb{Z})$ is contractible. By [Tah71], there are only two conjugacy classes of subgroups of order 2. They are represented by the subgroups generated by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The centralizer of A is isomorphic to $GL_2(\mathbb{Z})$ which is rationally acyclic since $SL_2(\mathbb{Z})$ is. The computation of $C\langle B \rangle$ is similar to the computation in Lemma 5.9. The matrix B has eigenvalues ± 1 and it is diagonalized over $\mathbb{Z}[\frac{1}{2}]$ by

$$Z^{-1} B Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

with

$$Z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad Z^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

The centralizer of the above diagonal matrix in $SL_2(\mathbb{Z}[\frac{1}{2}])$ is the subgroup of matrices of the form

$$C = \begin{pmatrix} b & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

with determinant equal to 1. Calculating ZCZ^{-1}, we get that a general element in the centralizer $C\langle B\rangle$ is given by a matrix of the form

$$\begin{pmatrix} a_{11} & a_{12}/2 & -a_{12}/2 \\ a_{21} & b/2 + a_{22}/2 & b/2 - a_{22}/2 \\ -a_{21} & b/2 - a_{22}/2 & b/2 + a_{22}/2 \end{pmatrix},$$

where a_{11}, a_{21}, $\pm b/2 + a_{22}/2$ are integers, a_{12} is an even integer, and $b(a_{11}a_{22} - a_{21}a_{12}) = 1$. This implies that b and a_{22} are also integers.

We recall that $\Gamma(2)$ denotes the congruence subgroup of $SL_2(\mathbb{Z})$ of matrices which are the identity mod 2. Using the extension

$$1 \longrightarrow \Gamma(2) \rtimes \mathbb{Z}/2 \longrightarrow GL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{F}_2) \longrightarrow 1$$

and the above calculation, we conclude that $C\langle B\rangle$ sits in an extension

$$1 \longrightarrow \Gamma(2) \rtimes \mathbb{Z}/2 \longrightarrow C\langle B\rangle \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

where $\mathbb{Z}/2 \leq SL_2(\mathbb{F}_2)$ can be thought as the subgroup of lower triangular matrices with determinant 1. By a Lyndon–Hochschild–Serre spectral sequence argument, it suffices to observe that $\Gamma(2) \rtimes \mathbb{Z}/2$ is rationally acyclic. This follows since $\Gamma(2) = F_2 \times \mathbb{Z}/2$, where F_2 is freely generated by the matrices $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, and the action of $\mathbb{Z}/2$ on $\Gamma(2)$ inverts both generators. The Lyndon–Hochschild–Serre spectral sequence now implies that $\Gamma(2) \rtimes \mathbb{Z}/2$ is rationally acyclic and this completes the proof. □

Remark 5.14. Using Lemma 5.13 and Corollary 2.5, as in the 3-primary case we get

$$\chi_{K(n)}(BSL_3(\mathbb{Z})) = \sum_{[g_1,\ldots,g_n] \in SL_3(\mathbb{Z})\backslash SL_3(\mathbb{Z})_{n,2}} \chi_{\mathbb{Q}}(BC\langle g_1,\ldots,g_n\rangle) = \sum_{[g_1,\ldots,g_n] \in SL_3(\mathbb{Z})\backslash SL_3(\mathbb{Z})_{n,2}} 1 = |SL_3(\mathbb{Z})\backslash SL_3(\mathbb{Z})_{n,2}|.$$

Now using the classification of finite subgroups of $SL_3(\mathbb{Z})$ and the explicit identification of conjugacy classes of 2-subgroups by [Tah71], one can directly compute $|SL_3(\mathbb{Z})\backslash SL_3(\mathbb{Z})_{n,2}| = 2^{2n+1} - 2^{n+1} + 1$ and recover the calculation in Proposition 5.12 proving that at the prime $p = 2$, we have

$$\chi_{K(n)}(BSL_3(\mathbb{Z})) = 2^{2n+1} - 2^{n+1} + 1.$$

We leave the details to the reader.

Finally, combining Corollary 3.5, Remark 5.14 and Proposition 5.12 as well as [Str98, Proposition 3.5], we obtain the following result:

Proposition 5.15. *Let $n \geq 1$ and $p = 2$. Then $E^*(BSL_3(\mathbb{Z}))$ at the prime 2 is even and free of rank $2^{2n+1} - 2^{n+1} + 1$ as a graded E^*-module.*

This proposition can also be deduced from [TY92, Sections 2–3].

5.3. **The special linear group $SL_2(\mathcal{O}_K)$ for a totally real field K.** We start by reviewing the case $SL_2(\mathbb{Z})$. It is well-known that $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$, see [Ser03, Example 4.2]. This decomposition can be used to easily compute invariants of $SL_2(\mathbb{Z})$. The classifying space $\underline{E}SL_2(\mathbb{Z})$ is a tree, compare Example 2.10; so we can apply the equivariant cellular structure of the Bass–Serre tree of $\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$ from [Ser03, Example 4.2] to compute

$$\begin{aligned}\chi_{\mathrm{orb}}(BSL_2(\mathbb{Z})) &= \chi_{\mathrm{orb}}(B\mathbb{Z}/6) + \chi_{\mathrm{orb}}(B\mathbb{Z}/4) - \chi_{\mathrm{orb}}(B\mathbb{Z}/2) \\ &= \tfrac{1}{6} + \tfrac{1}{4} - \tfrac{1}{2} \;=\; -\tfrac{1}{12}.\end{aligned}$$

To compute the classical Euler characteristic $\chi_{\mathbb{Q}}(BSL_2(\mathbb{Z}))$ one can use the Mayer–Vietoris sequence for the rational cohomology, and that finite groups are rationally acyclic. From this one obtains

$$\chi_{\mathbb{Q}}(BSL_2(\mathbb{Z})) = 1.$$

Similar Mayer–Vietoris arguments can be used to compute the E^*-cohomology and Morava K-theory of $BSL_2(\mathbb{Z})$, compare Proposition 2.9 and Example 2.10. At the prime $p = 2$, we get

$$\chi_{K(n)}(BSL_2(\mathbb{Z})) = 2^n + 4^n - 2^n = 4^n$$

and at the prime $p = 3$, we get

$$\chi_{K(n)}(BSL_2(\mathbb{Z})) = 3^n + 1 - 1 = 3^n.$$

The Mayer–Vietoris sequence shows that Morava K-theory and E-theory of $BSL_2(\mathbb{Z})$ are even and $E^*(BSL_2(\mathbb{Z}))$ is free as a graded E^*-module, of rank 4^n for $p = 2$, and of rank 3^n for $p = 3$.

The goal of this section is to generalize the above calculations to rings of integers in totally real fields. From now on we assume that K is a totally real field, e.g., $K = \mathbb{Q}(\sqrt{d})$, where d is any square-free positive integer. We let $\mathcal{O}_K$ denote the ring of integers of K. It follows from [Har71, p. 453] that

$$\chi_{\mathrm{orb}}(BSL_2(\mathcal{O}_K)) = \zeta_K(-1),$$

where ζ_K is the Dedekind zeta function of K. When $K = \mathbb{Q}$ this specializes to the Riemann zeta function ζ and one recovers the well-known identity from above

$$\chi_{\mathrm{orb}}(BSL_2(\mathbb{Z})) = -\tfrac{1}{12} = \zeta(-1).$$

Brown [Bro74, Lemma p. 251] shows that any finite subgroup $H \leq SL_2(\mathcal{O}_K)$ is cyclic; and if H is non-trivial and different from the center $\{\pm \mathrm{Id}\}$ of $SL_2(\mathcal{O}_K)$, then it is contained in the unique maximal finite subgroup. The maximal subgroup containing H is given by the centralizer $C(H)$ and additionally $C(H) = N(H)$, where $N(H)$ is the normalizer. Using this, Brown [Bro74, Section 9.1] proves the following formula for the rational Euler characteristic

$$\chi_{\mathbb{Q}}(BSL_2(\mathcal{O}_K)) = 2\zeta_K(-1) + \sum_{(H)} (1 - \tfrac{2}{|H|}),$$

where the sum runs over the conjugacy classes of the maximal finite subgroups. The number of such conjugacy classes for real quadratic fields is well understood in terms of class numbers of imaginary quadratic fields, see [Pre73] and [Hir73, p. 198].

The group $SL_2(\mathcal{O}_K)$ is an arithmetic group and hence admits a finite model for $\underline{E}G$ [Ji07, Theorem 3.2]. Using the behavior of finite subgroups, we can see that the conditions of [LW12, Corollary 2.8] are satisfied and we get the following homotopy pushout square in G-spaces, with $G = SL_2(\mathcal{O}_K)$:

$$\begin{tikzcd} \coprod_{(H)} G \times_H E(H/\{\pm \mathrm{Id}\}) \arrow[r] \arrow[d] & E(G/\{\pm \mathrm{Id}\}) \arrow[d] \\ \coprod_{(H)} G/H \arrow[r] & \underline{E}G, \end{tikzcd} \tag{5.16}$$

where the disjoint union runs over the conjugacy classes of the maximal finite subgroups. This pushout is obtained by applying [LW12, Corollary 2.8] to the inclusion of the families of subgroups $\{1, \{\pm \mathrm{Id}\}\} \subset \mathcal{F}in$. The following generalizes the above formula of Brown; for $n = 1$ it specializes to a result of Adem [Ade92, Example 4.4].

Proposition 5.17. *Let p be an odd prime, and K a totally real field. Then for any $n \geq 0$, we have*

$$\chi_{K(n)}(BSL_2(\mathcal{O}_K)) = 2\zeta_K(-1) + \sum_{(H)} \left(|H_{(p)}|^n - \tfrac{2}{|H|} \right),$$

where the sum runs over the conjugacy classes of the maximal finite subgroups.

Proof. Let $G = SL_2(\mathcal{O}_K)$. By taking quotients in the pushout (5.16), we get a homotopy pushout of spaces

$$\begin{tikzcd} \coprod_{(H)} B(H/\{\pm \mathrm{Id}\}) \arrow[r] \arrow[d] & B(G/\{\pm \mathrm{Id}\}) \arrow[d] \\ \coprod_{(H)} * \arrow[r] & \underline{B}G, \end{tikzcd}$$

where $\underline{B}G = G \backslash \underline{E}G$. Using the Mayer–Vietoris sequence of this pushout in Morava K-theory, we obtain the equation

$$\chi_{K(n)}(B(G/\{\pm \mathrm{Id}\})) + \sum_{(H)} 1 = \chi_{K(n)}(\underline{B}G) + \sum_{(H)} \chi_{K(n)}(B(H/\{\pm \mathrm{Id}\})).$$

Since p is odd, the Atiyah–Hirzebruch spectral sequence shows that $K(n)^*(B(G/\{\pm \mathrm{Id}\}))$ and $K(n)^*(B(H/\{\pm \mathrm{Id}\}))$ are isomorphic to $K(n)^*(BG)$ and $K(n)^*(BH)$, respectively. Additionally, since $\underline{B}G$ admits a finite CW-model, we get that $\chi_{K(n)}(\underline{B}G) = \chi_{\mathbb{Q}}(\underline{B}G) = \chi_{\mathbb{Q}}(BG)$. Hence the latter equation can be rewritten as

$$\chi_{K(n)}(BG) + \sum_{(H)} 1 = \chi_{\mathbb{Q}}(BG) + \sum_{(H)} \chi_{K(n)}(BH).$$

Using Brown's formula and $\chi_{K(n)}(BH) = |H_{(p)}|^n$ from [HKR00], we obtain the formula

$$\chi_{K(n)}(BSL_2(\mathcal{O}_K)) = 2\zeta_K(-1) + \sum_{(H)} \left(|H_{(p)}|^n - \tfrac{2}{|H|} \right).$$

□

Remark 5.18. There is an alternative way to prove Proposition 5.17 using Corollary 2.5. Indeed, by the classification of finite subgroups of $G = SL_2(\mathcal{O}_K)$ as given for example in [Bro74, Lemma p. 251],

$$|G \backslash G_{n,p}| - 1 = \sum_{(H)} (|H_{(p)}|^n - 1),$$

where the sum runs over the conjugacy classes of the maximal finite subgroups. This uses that for any non-trivial finite p-subgroup L, the normalizer $N(L)$ and the centralizer $C(L)$ agree and L is contained in the unique maximal finite subgroup. Using additionally that $C(L)$ is finite, by Corollary 2.5 and [Bro74, Section 9.1], we get

$$\begin{aligned}\chi_{K(n)}(BSL_2(\mathcal{O}_K)) &= \sum_{[g_1,\ldots,g_n]\in G\backslash G_{n,p}} \chi_{\mathbb{Q}}(BC\langle g_1,\ldots,g_n\rangle) \\ &= \chi_{\mathbb{Q}}(BSL_2(\mathcal{O}_K)) + |G\backslash G_{n,p}| - 1 \\ &= 2\zeta_K(-1) + \sum_{(H)}(1-\tfrac{2}{|H|}) + \sum_{(H)}(|H_{(p)}|^n - 1) \\ &= 2\zeta_K(-1) + \sum_{(H)}(|H_{(p)}|^n - \tfrac{2}{|H|}).\end{aligned}$$

Remark 5.19. Similar methods can be used to prove that at the prime $p = 2$, for any $n \geq 0$, we get the formula

$$\chi_{K(n)}(BSL_2(\mathcal{O}_K)) = 2^{n+1}\zeta_K(-1) + \sum_{(H)}\left(|H_{(2)}|^n - \tfrac{2^{n+1}}{|H|}\right).$$

The reason why the powers of 2 show up in this formula is that we have n-tuples $(\pm 1, \pm 1, \ldots, \pm 1) \in SL_2(\mathcal{O}_K)_{n,2}$. We will not give complete details for this calculation.

Now we consider the Morava E-cohomology of $BSL_2(\mathcal{O}_K)$; at height $n = 1$ this has already been done in [Ade92, Example 4.4]. Again we will deal with the odd primary case. For an odd prime p, Remark 5.18 and the character formula of Corollary 3.5 provide an isomorphism

$$L(E^*) \otimes_{E^*} E^*(BSL_2(\mathcal{O}_K)) \cong H^*(BSL_2(\mathcal{O}_K); L(E^*)) \oplus F,$$

where F is a free $L(E^*)$-module in even degrees of rank $\sum_{(H)}(|H_{(p)}|^n - 1)$, where (H) runs over the conjugacy classes of the maximal finite subgroups. Alternatively one could use the Mayer–Vietoris sequence associated to the pushout (5.16).

To compute $E^*(BSL_2(\mathcal{O}_K))$ integrally, it would help if $K(n)^*(BSL_2(\mathcal{O}_K))$ were even. This is the case when $K = \mathbb{Q}$ as discussed at the beginning of this section. We argue in the next example that this also holds for $K = \mathbb{Q}(\sqrt{5})$. Consequently, we get an integral computation of $E^*(BSL_2(\mathbb{Z}[\frac{1+\sqrt{5}}{2}]))$ for all odd primes and all heights.

Example 5.20. Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, where d is a square-free positive integer. In this case an infinite model for $\underline{E}G$ is given by $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the complex upper half plane. Here $SL_2(\mathcal{O}_K)$ acts through the embedding into $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. The latter acts on $\mathbb{H} \times \mathbb{H}$ via the Möbius transformations. It follows now from the Mayer–Vietoris sequence of the pushout (5.16), that if $K(n)^*(\underline{B}G)$ is even, then so is $K(n)^*(BG/\{\pm \mathrm{Id}\})$ and hence $K(n)^*(BG)$ will be even. It thus suffices to give a criterion which guarantees that $K(n)^*(\underline{B}G)$ is even.

The space $\underline{B}G = \mathbb{H} \times \mathbb{H}/SL_2(\mathcal{O}_K)$ is known as the Hilbert modular surface of the quadratic field K and its rational cohomology is well studied, see for example [Hir73]. Not many integral cohomology computations of the Hilbert modular surfaces are available, however. The thesis of Brownstein [Bro87] computes $H^*(\mathbb{H} \times \mathbb{H}/SL_2(\mathcal{O}_K), \mathbb{F}_p)$ for $K = \mathbb{Q}(\sqrt{5})$. It follows from the 0-line of the spectral sequence tabulated in [Bro87, p. 43] that for p an odd prime, the reduced cohomology $\widetilde{H}^n(\mathbb{H} \times \mathbb{H}/SL_2(\mathcal{O}_K), \mathbb{F}_p)$ vanishes unless $n = 2$. This implies that $H^*(\mathbb{H} \times \mathbb{H}/SL_2(\mathcal{O}_K), \mathbb{F}_p)$ is even when $K = \mathbb{Q}(\sqrt{5})$ and hence by the Atiyah–Hirzebruch spectral sequence we conclude that $K(n)^*(\underline{B}G) = K(n)^*(\mathbb{H} \times \mathbb{H}/SL_2(\mathcal{O}_K))$ is even. Thus $K(n)^*(BSL_2(\mathbb{Z}[\frac{1+\sqrt{5}}{2}]))$ is even.

By [Hir73, Theorem, p.198] and [Pre73], when $K = \mathbb{Q}(\sqrt{d})$, there are explicit formulas for computing the numbers and types of conjugacy classes of maximal finite subgroups of $SL_2(\mathcal{O}_K)$. If $d \geq 7$ is square-free and coprime to 6, then any maximal finite subgroup of $SL_2(\mathcal{O}_K)$ is isomorphic to $\mathbb{Z}/4$ or $\mathbb{Z}/6$ and the number of conjugacy classes of such subgroups is explicitly determined by class numbers of imaginary quadratic fields with discriminants being explicit multiples of $-d$. For $d = 5$ case by [Hir73, Table, p. 200], it follows that any maximal finite subgroup of $SL_2(\mathbb{Z}[\frac{1+\sqrt{5}}{2}])$ is isomorphic to $\mathbb{Z}/4$, $\mathbb{Z}/6$ or $\mathbb{Z}/10$ and there are exactly two conjugacy classes for each of these groups. We also know the special value of the Dedekind zeta function $\zeta_{\mathbb{Q}(\sqrt{5})}(-1) = \frac{1}{30}$. This allows to make the formula of Proposition 5.17 more explicit. For example, for $p = 3$ we have

$$\begin{aligned}\chi_{K(n)}(BSL_2(\mathbb{Z}[\tfrac{1+\sqrt{5}}{2}])) &= \tfrac{1}{15} + 2(3^n - \tfrac{2}{6}) + 2(1 - \tfrac{2}{4}) + 2(1 - \tfrac{2}{10}) \\ &= 2 \cdot 3^n + 2.\end{aligned}$$

Since $K(n)^*(BSL_2(\mathbb{Z}[\frac{1+\sqrt{5}}{2}]))$ is even, by [Str98, Proposition 3.5] we conclude that the E^*-module $E^*(BSL_2(\mathbb{Z}[\frac{1+\sqrt{5}}{2}]))$ is even and free of rank $2 \cdot 3^n + 2$. Similar results can be obtained at the prime $p = 5$.

5.4. **Crystallographic groups.** In this subsection we consider $\chi_{K(n)}(BG)$ for any group G which fits into a short exact sequence ($n \geq 1$)

$$1 \longrightarrow \mathbb{Z}^m \longrightarrow G \longrightarrow \mathbb{Z}/p \longrightarrow 1.$$

Such groups are certain crystallographic groups and their invariants have interesting behavior depending on the action of $\mathbb{Z}/p$ on $\mathbb{Z}^m$. The rational and orbifold Euler characteristics were studied in [Lüc02b, Example 6.94] and the homological and K-theoretic computations have been done in [Ade87, AGPP08, DL13]. Using these calculations and analysis of centralizers by Corollary 3.5 and Corollary 2.5, one can give explicit formulas for $L(E^*) \otimes_{E^*} E^*(BG)$ and $\chi_{K(n)}(BG)$. In this subsection we will mostly focus on $\chi_{K(n)}(BG)$ and only briefly mention the calculation of $L(E^*) \otimes_{E^*} E^*(BG)$ in the special case of free actions.

Crystallographic groups G in the above sense admit a finite G-CW model for $\underline{E}G$ by [Lüc02b, Example 6.94]. If G is torsion-free, then $\chi_{K(n)}(BG) = \chi_{\mathbb{Q}}(BG) = 0$. If G has torsion, then it has to be p-torsion and any finite subgroup of G is isomorphic to $\mathbb{Z}/p$. In this case we have an isomorphism $G \cong \mathbb{Z}^m \rtimes_\rho \mathbb{Z}/p$, where ρ is an integral representation of $\mathbb{Z}/p$. If the action by $\mathbb{Z}/p$ has non-trivial fixed points, then for any $g \in G$ of order p, the centralizer $C\langle g \rangle$ is isomorphic to the product of $\mathbb{Z}/p$ with $\mathbb{Z}^l$, for a fixed $l \geq 1$ (in fact $\mathbb{Z}^l = (\mathbb{Z}^m)^{\mathbb{Z}/p}$) and hence $\chi_{\mathbb{Q}}(BC\langle g \rangle) = 0$. It follows from [Lüc02b, Example 6.94] that $\chi_{\mathbb{Q}}(BG) = 0$; alternatively, one can also use Brown's formula [Bro82] from Example 4.8. Hence using Corollary 2.5, we get $\chi_{K(n)}(BG) = 0$ at the prime p.

Finally, we suppose $\mathbb{Z}/p$ acts freely on $\mathbb{Z}^m \setminus \{0\}$. Let r denote the number of conjugacy classes of non-trivial finite subgroups. In this case every non-trivial finite subgroup of G is self-normalizing, and hence

$$\chi_{K(n)}(BG) = \chi_{\mathbb{Q}}(BG) + r(p^n - 1)$$

by Corollary 2.5. Using [Lüc02b, Example 6.94] or [Bro82], we know that $\chi_{\mathbb{Q}}(BG) = -\frac{r}{p} + r$. Hence we obtain the formula

$$\chi_{K(n)}(BG) = rp^n - \tfrac{r}{p}.$$

It follows from the computations in [Ade87, AGPP08, DL13] that in fact if $\mathbb{Z}/p$ acts freely on $\mathbb{Z}^m \setminus \{0\}$, then we have $m = k(p-1)$ and $r = p^k$. Finally, we observe that analogously to the case of right angled Coxeter groups in Example 5.6, it also makes sense to consider the case $n = -1$ in the above formula, which recovers $\chi_{\mathrm{orb}}(BG) = 0$.

We conclude by observing that when $\mathbb{Z}/p$ acts freely on $\mathbb{Z}^m \setminus \{0\}$, then Corollary 3.5 provides an isomorphism

$$L(E^*) \otimes_{E^*} E^*(BG) \cong H^*(BG; L(E^*)) \oplus F,$$

where F is a free $L(E^*)$-module in even degrees of rank $r(p^n - 1)$. The rational cohomology $H^*(BG; \mathbb{Q})$ is known by [Ade87, AGPP08, DL13]. Thus one obtains the full computation of $L(E^*) \otimes_{E^*} E^*(BG)$. We do not go here into more details.

5.5. **The general linear group $GL_{p-1}(\mathbb{Z})$ for a prime $p \geq 5$.** The examples discussed so far are computable in two different ways, either by Corollary 2.5 and Corollary 3.5, or by use of explicit cellular models for $\underline{E}G$ and arguments as in Proposition 2.9 and Proposition 5.17.

In this section we present a first example $G = GL_{p-1}(\mathbb{Z})$ where it is essential to use our formulas from Corollary 2.5 and Corollary 3.5, i.e., we do not know other ways to arrive at these computations. At the height $n = 1$ these calculations were already done by Adem [Ade92, Example 4.5]. As an arithmetic group, $GL_{p-1}(\mathbb{Z})$ has a finite model for $\underline{E}G$ see [Ji07, Theorem 3.2]; we do not know any sufficiently explicit cellular structure to work with.

Proposition 5.21. *Let $p \geq 5$ be a prime. Then for every $n \geq 0$,*

$$\chi_{K(n)}(BGL_{p-1}(\mathbb{Z})) = \chi_{\mathbb{Q}}(BGL_{p-1}(\mathbb{Z})).$$

Proof. It follows from [LN98, Proposition 1.1] that the only p-power order elements in $GL_{p-1}(\mathbb{Z})$ are elements of order p. Every elementary abelian p-subgroup in $GL_{p-1}(\mathbb{Z})$ has the rank at most 1, see e.g., [Ash89, Section 2]; so every finite p-subgroup of $GL_{p-1}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/p$. It follows from [Ash89, Lemma 4] that $C\langle A\rangle$ for any A of order p is isomorphic to the group of units in $\mathbb{Z}[\zeta_p]$ which by the Dirichlet unit theorem is isomorphic to

$$\mathbb{Z}/p \times \mathbb{Z}/2 \times \mathbb{Z}^{\frac{p-3}{2}}.$$

Corollary 2.5 yields

$$\chi_{K(n)}(BGL_{p-1}(\mathbb{Z})) = \chi_{\mathbb{Q}}(BGL_{p-1}(\mathbb{Z})) + \sum_{[A_1,\ldots,A_n]} \chi_{\mathbb{Q}}(BC\langle A_1, \ldots, A_n\rangle),$$

where $[A_1, \ldots, A_n] \in GL_{p-1}(\mathbb{Z})\backslash(GL_{p-1}(\mathbb{Z})_{n,p} - \{(1, \ldots, 1)\})$. By the above observation

$$C\langle A_1, \ldots, A_n\rangle \cong \mathbb{Z}/p \times \mathbb{Z}/2 \times \mathbb{Z}^{\frac{p-3}{2}}$$

and hence $\chi_{\mathbb{Q}}(BC\langle A_1, \ldots, A_n\rangle) = 0$ since the rational Euler characteristic of tori is zero. This gives the desired result. $\square$

For Morava E-theory we obtain:

Proposition 5.22. *Let $p \geq 5$ be a prime. Then the character map gives an isomorphism*

$$\begin{aligned} &L(E^*) \otimes_{E^*} E^*(BGL_{p-1}(\mathbb{Z})) \cong \\ &\quad H^*(BGL_{p-1}(\mathbb{Z}); L(E^*)) \oplus \bigoplus_{\frac{p^n-1}{p-1}|\operatorname{Cl}(\mathbb{Q}(\zeta_p))|} (L(E^*) \oplus L(E^*)[1])^{\otimes_{L(E^*)} \frac{p-3}{2}}, \end{aligned}$$

where $\operatorname{Cl}(\mathbb{Q}(\zeta_p))$ denotes the ideal class group of $\mathbb{Q}(\zeta_p)$.

Proof. Given $A \in GL_{p-1}(\mathbb{Z})$ of order p, then $\zeta_p = e^{2\pi i/p}$ is one of its eigenvalues. Let $x = (x_1, \dots, x_{p-1}) \in \mathbb{Q}(\zeta_p)^{p-1}$ be one of the eigenvectors and $I(A)$ denote the ideal class of the fractional ideal $\mathbb{Z}x_1 + \cdots + \mathbb{Z}x_{p-1}$ of $\mathbb{Q}(\zeta_p)$. This class is independent of the choice of x since the eigenspaces of A over $\mathbb{Q}(\zeta_p)$ are 1-dimensional. The latter follows since the Galois group $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)\colon \mathbb{Q})$ is isomorphic to $\mathbb{Z}/p^\times \cong \mathbb{Z}/(p-1)$ and all Galois conjugates of $\zeta_p = e^{2\pi i/p}$ are eigenvalues of A. In fact, the ideal class $I(A)$ only depends on the conjugacy class of A and by [LM33], the map

$$I : GL_{p-1}(\mathbb{Z})\backslash GL_{p-1}(\mathbb{Z})_{1,p} \longrightarrow \mathrm{Cl}(\mathbb{Q}(\zeta_p))$$

from conjugacy classes of the order p-elements to the ideal class group is a bijection; see also [SY97] and [Ash89]. The Galois group $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)\colon \mathbb{Q})$ acts on the class group $\mathrm{Cl}(\mathbb{Q}(\zeta_p))$ by the Galois action. And Δ acts on $GL_{p-1}(\mathbb{Z})\backslash GL_{p-1}(\mathbb{Z})_{1,p}$ by taking powers of the matrices. By [Ash89, Section 1], the isomorphism I is in fact Δ-equivariant with respect to these two actions. This implies that the conjugacy classes of order p subgroups are in bijection with the set of orbits $\Delta\backslash \mathrm{Cl}(\mathbb{Q}(\zeta_p))$. Additionally, it follows as in [Ash89, Section 1] that for any order p element A, there is a short exact sequence

$$1 \longrightarrow C\langle A\rangle \longrightarrow N\langle A\rangle \longrightarrow S_{I(A)} \longrightarrow 1,$$

where $S_{I(A)} \leq \Delta$ is the stabilizer of $I(A)$ (which agrees with the stabilizer of the conjugacy class of A), the group $N\langle A\rangle$ is the normalizer of the subgroup generated by A, and $C\langle A\rangle$ is the centralizer of A. Any tuple $(A_1, \dots, A_n) \in GL_{p-1}(\mathbb{Z})_{n,p}$ with at least one non-trivial element is contained in the unique subgroup of order p. This implies that

$$|GL_{p-1}(\mathbb{Z})\backslash GL_{p-1}(\mathbb{Z})_{n,p}| - 1 = \sum_{(H)} \frac{|H|^n - 1}{|N(H)/C(H)|}, \tag{5.23}$$

where H runs over conjugacy classes of subgroups of order p, and $N(H)$ is the normalizer of H and $C(H)$ is the centralizer of H. This and the above discussion imply that

$$|GL_{p-1}(\mathbb{Z})\backslash GL_{p-1}(\mathbb{Z})_{n,p}| - 1 = \sum_{[I]\in\Delta\backslash \mathrm{Cl}(\mathbb{Q}(\zeta_p))} \frac{p^n - 1}{|S_I|},$$

where $S_I \leq \Delta$ is the stabilizer of $I \in \mathrm{Cl}(\mathbb{Q}(\zeta_p))$. Let O_I, denote the orbit of I. Then

$$\begin{aligned} |GL_{p-1}(\mathbb{Z})\backslash GL_{p-1}(\mathbb{Z})_{n,p}| - 1 &= \sum_{[I]\in\Delta\backslash \mathrm{Cl}(\mathbb{Q}(\zeta_p))} \tfrac{p^n-1}{|S_I|\cdot|O_I|} \cdot |O_I| \\ &= \sum_{[I]\in\Delta\backslash \mathrm{Cl}(\mathbb{Q}(\zeta_p))} \tfrac{p^n-1}{p-1} \cdot |O_I| \\ &= \tfrac{p^n-1}{p-1} \sum_{[I]\in\Delta\backslash \mathrm{Cl}(\mathbb{Q}(\zeta_p))} |O_I| = \tfrac{p^n-1}{p-1} |\mathrm{Cl}(\mathbb{Q}(\zeta_p))|. \end{aligned}$$

The last identity is the class equation.

Now as observed in the proof of Proposition 5.21, for any tuple $(A_1, \dots, A_n) \in GL_{p-1}(\mathbb{Z})_{n,p} - \{(1, \dots, 1)\}$, we have an isomorphism

$$C\langle A_1, \dots, A_n\rangle \cong \mathbb{Z}/p \times \mathbb{Z}/2 \times \mathbb{Z}^{\frac{p-3}{2}}.$$

Hence, one obtains

$$\begin{aligned} H^*(BC\langle A_1, \dots, A_n\rangle; L(E^*)) &\cong H^*((S^1)^{\times \frac{p-3}{2}}; L(E^*)) \\ &\cong (L(E^*) \oplus L(E^*)[1])^{\otimes_{L(E^*)} \frac{p-3}{2}}. \end{aligned}$$

This together with the previous paragraph completes the proof. $\square$

Remark 5.24. To compute $\chi_{K(n)}(BGL_2(\mathbb{Z}))$ and $E^*(BGL_2(\mathbb{Z}))$ at the prime $p = 3$, one can proceed in much the same way as for $p \geq 5$, with the caveat that in this case the centralizers are finite. Alternatively, one can also proceed directly using the calculations for $SL_2(\mathbb{Z})$ at the beginning of Subsection 5.3 and the extension

$$1 \longrightarrow SL_2(\mathbb{Z}) \longrightarrow GL_2(\mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

It follows that $\chi_{K(n)}(BGL_2(\mathbb{Z})) = (3^n + 1)/2$ and $E^*(BGL_2(\mathbb{Z}))$ is even and free of rank $(3^n + 1)/2$ as a graded E^*-module.

Remark 5.25. The rational Euler characteristic of $BGL_{p-1}(\mathbb{Z})$ has been computed. For $p \geq 13$ we have $\chi_{\mathbb{Q}}(BGL_{p-1}(\mathbb{Z})) = 0$ by [Hor05, Theorem 0.1 (a)]. Moreover, $\chi_{\mathbb{Q}}(BGL_4(\mathbb{Z})) = \chi_{\mathbb{Q}}(BGL_6(\mathbb{Z})) = \chi_{\mathbb{Q}}(BGL_{10}(\mathbb{Z})) = 1$ by the following explicit results: From [DEKM19, pages 16-17] and [LS78] we know that $GL_4(\mathbb{Z})$ is rationally acyclic. Rational cohomology of $GL_6(\mathbb{Z})$ is computed in [EVGS13, Theorem 7.3]:

$$H^n(GL_6(\mathbb{Z}); \mathbb{Q}) = \begin{cases} \mathbb{Q}, \ n = 0, 5, 8 \\ 0, \ \text{otherwise}, \end{cases}$$

implying that $\chi_{\mathbb{Q}}(BGL_6(\mathbb{Z})) = 1$. Horozov in [Hor05, Theorem 3.3] shows that $\chi_{\mathbb{Q}}(BGL_{10}(\mathbb{Z})) = 1$. To our knowledge $H^n(GL_{p-1}(\mathbb{Z}); \mathbb{Q})$ is not computed for $p \geq 11$. Hence Proposition 5.22 only gives a full computation when $p = 5, 7$.

5.6. **The special linear group $SL_{p-1}(\mathbb{Z})$ for a prime $p \geq 5$.** The group $GL_{p-1}(\mathbb{Z})$ non-canonically splits as a semidirect product of $SL_{p-1}(\mathbb{Z})$ and $\mathbb{Z}/2$; one can use this to obtain formulas for the Morava K-theory Euler characteristic and E^*-cohomology of $BSL_{p-1}(\mathbb{Z})$ from those for $BGL_{p-1}(\mathbb{Z})$ from Subsection 5.5. Instead of exploiting the splitting, we prefer to follow the same arguments as in the previous subsection and show that computations for $SL_{p-1}(\mathbb{Z})$ can be independently deduced. We recall that $SL_{p-1}(\mathbb{Z})$ has finite model of $\underline{E}G$ as an arithmetic group [Ji07, Theorem 3.2].

Proposition 5.26. *Let $p \geq 5$ be a prime.*

(i) *For any integer $n \geq 0$,*

$$\chi_{K(n)}(BSL_{p-1}(\mathbb{Z})) = \chi_{\mathbb{Q}}(BSL_{p-1}(\mathbb{Z})).$$

(ii) *The character map gives an isomorphism*

$$L(E^*) \otimes_{E^*} E^*(BSL_{p-1}(\mathbb{Z})) \cong$$
$$H^*(BSL_{p-1}(\mathbb{Z}); L(E^*)) \oplus \bigoplus_{2 \cdot \frac{p^n-1}{p-1} |\operatorname{Cl}(\mathbb{Q}(\zeta_p))|} (L(E^*) \oplus L(E^*)[1])^{\otimes_{L(E^*)} \frac{p-3}{2}},$$

Proof. Given an integral ideal I of $\mathbb{Q}(\zeta_p)$ and two bases $(x_1, \ldots, x_{p-1})$ and $(y_1, \ldots, y_{p-1})$ for I (i.e., $I = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_{p-1} = \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_{p-1}$), we say that $(x_1, \ldots, x_{p-1})$ and $(y_1, \ldots, y_{p-1})$ have the same orientation if there exists $C \in SL_{p-1}(\mathbb{Z})$ such that

$$C \begin{pmatrix} x_1 \\ \vdots \\ x_{p-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{p-1} \end{pmatrix}.$$

Let $\operatorname{Cl}(\mathbb{Q}(\zeta_p))^+$ denote the set of oriented ideal classes. An element of $\operatorname{Cl}(\mathbb{Q}(\zeta_p))^+$ is an equivalence class of pairs $(I, [x_1, \ldots, x_{p-1}])$, where I is an integral ideal and $[x_1, \ldots, x_{p-1}]$ an orientation class of a basis. A pair $(I, [x_1, \ldots, x_{p-1}])$ is equivalent to $(J, [y_1, \ldots, y_{p-1}])$ if and only if there exist $r, s \in \mathbb{Z}[\zeta_p] \setminus \{0\}$ such that $rI = sJ$ and $(rx_1, \ldots, rx_n)$ and $(sy_1, \ldots, sy_n)$ have the same orientation. The bijection I :

$GL_{p-1}(\mathbb{Z})\backslash GL_{p-1}(\mathbb{Z})_{1,p} \longrightarrow \mathrm{Cl}(\mathbb{Q}(\zeta_p))$ from the proof of Proposition 5.22 induces a well-defined bijection

$$I^+ : SL_{p-1}(\mathbb{Z})\backslash SL_{p-1}(\mathbb{Z})_{1,p} \longrightarrow \mathrm{Cl}(\mathbb{Q}(\zeta_p))^+$$

sending a class of A to the class of $(\mathbb{Z}x_1 + \cdots + \mathbb{Z}x_{p-1}, [x_1, \ldots, x_{p-1}])$, where $(x_1, \ldots, x_{p-1})$ is an eigenvector of A with respect to the eigenvalue ζ_p (recall $I(A)$ is the class of the fractional ideal $\mathbb{Z}x_1 + \cdots + \mathbb{Z}x_{p-1}$). Since every fractional ideal has exactly two equivalence classes of bases it follows that $|\mathrm{Cl}(\mathbb{Q}(\zeta_p))^+| = 2|\mathrm{Cl}(\mathbb{Q}(\zeta_p))|$. By [Ash89, Lemma 4] the centralizer of A in $GL_{p-1}(\mathbb{Z})$ is generated by

$$\{-\mathrm{Id},\ A,\ \tfrac{A^k-1}{A-1} \mid 1 \le i \le p-1,\ \ 2 \le k < p/2\}.$$

All these matrices are in fact elements of $SL_{p-1}(\mathbb{Z})$, so that the centralizers of A in $GL_{p-1}(\mathbb{Z})$ and $SL_{p-1}(\mathbb{Z})$ agree. The rest follows exactly as in Subsection 5.5 for $GL_{p-1}(\mathbb{Z})$, using the action of the Galois group $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)\colon \mathbb{Q})$ on $\mathrm{Cl}(\mathbb{Q}(\zeta_p))^+$ and a formula analogous to (5.23). □

Remark 5.27. By [Hor05], we have $\chi_{\mathbb{Q}}(BSL_{p-1}(\mathbb{Z})) = 0$, for $p \ge 13$. The paper [LS78] computes the rational cohomology

$$H^n(SL_4(\mathbb{Z}); \mathbb{Q}) = \begin{cases} \mathbb{Q},\ n = 0, 3, \\ 0,\ \text{otherwise,} \end{cases}$$

implying that $\chi_{\mathbb{Q}}(BSL_4(\mathbb{Z})) = 0$. Next, the rational cohomology for $SL_6(\mathbb{Z})$ is computed in [EVGS13]:

$$H^n(SL_6(\mathbb{Z}); \mathbb{Q}) = \begin{cases} \mathbb{Q} \oplus \mathbb{Q},\ n = 5, \\ \mathbb{Q},\ n = 0, 8, 9, 10, \\ 0,\ \text{otherwise.} \end{cases}$$

This implies $\chi_{\mathbb{Q}}(BSL_6(\mathbb{Z})) = 0$. We are not aware of a calculation of $\chi_{\mathbb{Q}}(BSL_{10}(\mathbb{Z}))$, and to our knowledge $H^n(SL_{p-1}(\mathbb{Z}); \mathbb{Q})$ has not been computed for $p \ge 11$. Hence Proposition 5.26 only gives a full computation of $L(E^*) \otimes_{E^*} E^*(BSL_{p-1}(\mathbb{Z}))$ when $p = 5, 7$.

5.7. **The symplectic group $Sp_{p-1}(\mathbb{Z})$ for a prime $p \ge 5$.** In this subsection we apply the main results of this paper to the symplectic group $Sp_{p-1}(\mathbb{Z})$. The main references for the conjugacy classification of p-subgroups of $Sp_{p-1}(\mathbb{Z})$ are [Bus02] and [SY97]. The arguments are very analogous to those in the previous two subsections. Again, $Sp_{p-1}(\mathbb{Z})$ has finite model of $\underline{E}G$ as an arithmetic group [Ji07, Theorem 3.2].

Proposition 5.28. *Let $p \ge 5$ be a prime.*

(i) *For any integer $n \ge 0$,*

$$\chi_{K(n)}(BSp_{p-1}(\mathbb{Z})) = \chi_{\mathbb{Q}}(BSp_{p-1}(\mathbb{Z})) + 2^{\frac{p-1}{2}} \cdot h_p^- \cdot \tfrac{p^n-1}{p-1}.$$

(ii) *The character map gives an isomorphism*

$$L(E^*) \otimes_{E^*} E^*(BSp_{p-1}(\mathbb{Z})) \cong H^*(BSp_{p-1}(\mathbb{Z}); L(E^*)) \oplus F,$$

where F is a free $L(E^*)$-module in even degrees of rank $2^{\frac{p-1}{2}} \cdot h_p^- \cdot \frac{p^n-1}{p-1}$.

Here

$$h_p^- = \frac{|\mathrm{Cl}(\mathbb{Q}(\zeta_p)|}{|\mathrm{Cl}(\mathbb{Q}(\zeta_p + \zeta_p^{-1})|}$$

is the relative class number.

Proof. Since $Sp_{p-1}(\mathbb{Z}) \leq SL_{p-1}(\mathbb{Z})$, we know that any non-trivial p-subgroup of $Sp_{p-1}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/p$. Using a similar construction as in the proof of Proposition 5.22, the paper [SY97] shows that the conjugacy classes of the elements of order p in $Sp_{p-1}(\mathbb{Z})$ biject with the set of equivalence classes of pairs (I, a), where I is an integral ideal and $I \cdot \overline{I} = (a)$, where $a = \overline{a}$. It follows form [SY97, Theorem 3] that the number of such equivalence classes is equal to $2^{\frac{p-1}{2}} h_p^-$. By [Bus02, Section 3.2] and [Bro74], we know that for any element A of order p, the centralizer of A is finite and

$$C\langle A\rangle \cong \mathbb{Z}/p \times \mathbb{Z}/2.$$

The Galois group $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)\colon \mathbb{Q})$ acts on the set of equivalence classes of pairs (I, a) and very similar arguments as in Subsection 5.5, including a formula analogous to (5.23), imply that the cardinality of the set of equivalence classes $Sp_{p-1}(\mathbb{Z})\backslash(Sp_{p-1}(\mathbb{Z})_{n,p} - \{(1, \dots, 1)\})$ is equal to the number $\frac{p^n-1}{p-1} \cdot 2^{\frac{p-1}{2}} \cdot h_p^-$. $\square$

Remark 5.29. The paper [BL92] computes the rational cohomology

$$H^n(Sp_4(\mathbb{Z}); \mathbb{Q}) = \begin{cases} \mathbb{Q}, \ n = 0, 2, \\ 0, \ \text{otherwise}, \end{cases}$$

implying that $\chi_{\mathbb{Q}}(BSp_4(\mathbb{Z})) = 2$. Next, the rational cohomology for $Sp_6(\mathbb{Z})$ is computed in [Hai02]

$$H^n(Sp_6(\mathbb{Z}); \mathbb{Q}) = \begin{cases} \mathbb{Q}, \ n = 0, 2, 4, \\ \mathbb{Q} \oplus \mathbb{Q}, \ n = 6, \\ 0, \ \text{otherwise}. \end{cases}$$

This implies $\chi_{\mathbb{Q}}(BSp_6(\mathbb{Z})) = 5$. We are not aware of a calculation of $H^n(Sp_{p-1}(\mathbb{Z}); \mathbb{Q})$ for $p \geq 11$. Hence Proposition 5.28 only gives a full computation of $L(E^*) \otimes_{E^*} E^*(BSL_{p-1}(\mathbb{Z}))$ when $p = 5, 7$.

To our knowledge $\chi_{\mathbb{Q}}(BSp_{2l}(\mathbb{Z}))$ is only known for $l \leq 9$, see [HT18, Appendix]. For example when $p = 19$, we have $\chi_{\mathbb{Q}}(BSp_{18}(\mathbb{Z})) = 528$. The class number of the cyclotomic field $\mathbb{Q}(\zeta_n)$ is equal to 1 for $n \leq 22$. Hence, by Proposition 5.28, we get

$$\chi_{K(n)}(BSp_{18}(\mathbb{Z})) = 528 + 2^9 \cdot \tfrac{19^n-1}{18} = \tfrac{256 \cdot 19^n + 4496}{9}$$

at the prime $p = 19$. Computations for lower primes are done similarly using a list for $\chi_{\mathbb{Q}}(BSp_{2l}(\mathbb{Z}))$ in [HT18, Appendix].

5.8. **The mapping class group $\Gamma_{\frac{p-1}{2}}$ for a prime $p \geq 5$.** Let $\Gamma_{\frac{p-1}{2}}$ be the mapping class group of the closed oriented surface of genus $(p-1)/2$. The papers [Bro90, JW10, Mis10] show that mapping class groups admit a finite model for $\underline{E}G$. The calculations in this subsection are based on results from Chapter III of Xia's thesis [Xia92].

It follows from [Xia92, Chapter III] that any non-trivial finite p-subgroup of $\Gamma_{\frac{p-1}{2}}$ is isomorphic to $\mathbb{Z}/p$. One considers two cases:

Case 1. $p = 6k - 1$: By [Xia92, Corollary 3.3.2] the number of conjugacy classes of non-trivial p-subgroups is equal to $(p+1)/6$, for any such subgroup H, the normalizer $N(H)$ is finite and we have $N(H)/C(H) = 1$. Hence, using a formula analogous to (5.23), we obtain

$$|\Gamma_{\frac{p-1}{2}} \backslash (\Gamma_{\frac{p-1}{2}})_{n,p}| = \tfrac{p+1}{6}(p^n - 1) + 1.$$

Case 2. $p = 6k + 1$: By [Xia92, Corollary 3.3.2] the number of conjugacy classes of non-trivial p-subgroups is equal to $(p+5)/6$. For any such subgroup H, the normalizer $N(H)$ is finite. For one conjugacy class we have $N(H)/C(H) = 3$ and

$N(H)/C(H) = 1$ for all the others. Hence, again using a similar argument as in Subsection 5.5, we obtain

$$|\Gamma_{\frac{p-1}{2}} \backslash (\Gamma_{\frac{p-1}{2}})_{n,p}| = (\tfrac{p+5}{6} - 1)(p^n - 1) + \tfrac{p^n-1}{3} + 1 = \tfrac{p+1}{6}(p^n - 1) + 1.$$

Combining these results with Corollary 2.5 and Corollary 3.5, we obtain

Proposition 5.30. *Let $p \geq 5$ be a prime.*

(i) *For any integer $n \geq 0$,*

$$\chi_{K(n)}(B\Gamma_{\frac{p-1}{2}}) = \chi_{\mathbb{Q}}(B\Gamma_{\frac{p-1}{2}}) + \tfrac{(p^n-1)(p+1)}{6}.$$

(ii) *The character map gives an isomorphism*

$$L(E^*) \otimes_{E^*} E^*(B\Gamma_{\frac{p-1}{2}}) \cong H^*(B\Gamma_{\frac{p-1}{2}}; L(E^*)) \oplus F,$$

where F is a free $L(E^*)$-module in even degrees of rank $(p^n - 1)(p + 1)/6$.

Remark 5.31. In the case $p = 5$, the rational cohomology of Γ_2 is well known, see for example [Kaw97]:

$$H^n(\Gamma_2; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & n = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Next, the rational cohomology for Γ_3 is computed in [Loo93]

$$H^n(\Gamma_3; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & n = 0, 1, 6, \\ 0, & \text{otherwise}. \end{cases}$$

Hence Proposition 5.30 gives a full computation of $L(E^*) \otimes_{E^*} E^*(B\Gamma_{\frac{p-1}{2}})$ when $p = 5, 7$.

Harer and Zagier in [HZ86] give a general formula for $\chi_{\mathbb{Q}}(\Gamma_g)$ for any genus g. For example, when $p = 31$, we have $\chi_{\mathbb{Q}}(\Gamma_{15}) = 717766$ and hence

$$\chi_{K(n)}(B\Gamma_{15}) = 717766 + \tfrac{16(31^n-1)}{3} = \tfrac{16\cdot 31^n+2153282}{3}.$$

For lower primes we get similar formulas using the table in [HZ86, §6].

References

[Ade87] A Adem, *$\mathbb{Z}/p\mathbb{Z}$ actions on $(S^n)^k$*, Trans. Amer. Math. Soc. **300** (1987), no. 2, 791–809.

[Ade92] ———, *On the K-theory of the classifying space of a discrete group*, Math. Ann. **292** (1992), no. 2, 319–327.

[AGPP08] A Adem, J Ge, J Pan, and N Petrosyan, *Compatible actions and cohomology of crystallographic groups*, J. Algebra **320** (2008), no. 1, 341–353.

[AS69] M F Atiyah and G B Segal, *Equivariant K-theory and completion*, J. Differential Geometry **3** (1969), 1–18.

[Ash89] A Ash, *Farrell cohomology of* $\mathrm{GL}(n, \mathbb{Z})$, Israel J. Math. **67** (1989), no. 3, 327–336.

[Bar07] D A Barkauskas, *Centralizers in graph products of groups*, J. Algebra **312** (2007), no. 1, 9–32.

[BS73] A Borel and J-P Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491.

[BK72] A K Bousfield, D M Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972. v+348 pp.

[Bro90] S A Broughton, *The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups*, Topology Appl. **37** (1990), no. 2, 101–113.

[Bro74] K S Brown, *Euler characteristics of discrete groups and G-spaces*, Invent. Math. **27** (1974), 229–264.

[Bro82] ———, *Complete Euler characteristics and fixed-point theory*, J. Pure Appl. Algebra **24** (1982), no. 2, 103–121.

[Bro87] A B Brownstein, *Homology of Hilbert modular groups*, ProQuest LLC, Ann Arbor, MI, 1987, Thesis (Ph.D.)–University of Michigan.

[BL92] A Brownstein and R Lee, *Cohomology of the symplectic group* $\mathrm{Sp}_4(\mathbb{Z})$. *I. The odd torsion case*, Trans. Amer. Math. Soc. **334** (1992), no. 2, 575–596.

[Bus02] C Busch, *The Farrell cohomology of* $\mathrm{Sp}(p-1,\mathbb{Z})$, Doc. Math. **7** (2002), 239–254.

[Dav15] M W Davis, *The geometry and topology of Coxeter groups*, Introduction to modern mathematics, Adv. Lect. Math. (ALM), vol. 33, Int. Press, Somerville, MA, 2015, pp. 129–142.

[DL13] J F Davis and W Lück, *The topological K-theory of certain crystallographic groups*, J. Noncommut. Geom. **7** (2013), no. 2, 373–431.

[DL17] D Degrijse and I J Leary, *Equivariant vector bundles over classifying spaces for proper actions*, Algebr. Geom. Topol. **17** (2017), no. 1, 131–156.

[DEKM19] M Dutou Sikrić, P Elbaz-Vincent, A Kupers, and J Martinet, *Voronoi complexes in higher dimensions, cohomology of* $GL_N(\mathbb{Z})$ *for* $n \geq 8$ *and the triviality of* $K_8(\mathbb{Z})$, `arXiv:1910.11598`, 2019.

[EVGS13] P Elbaz-Vincent, H Gangl, and C Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math. **245** (2013), 587–624.

[GH07] D Gepner and A Henriques, *Homotopy theory of orbispaces*, `arXiv:math/0701916`, 2007.

[GH04] P G Goerss, M J Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, 151–200, London Math. Soc. Lecture Note Ser., 315, Cambridge Univ. Press, Cambridge, 2004.

[GLO15] M Grant, G Lupton, and J Oprea, *New lower bounds for the topological complexity of aspherical spaces*, Topology Appl. **189** (2015), 78–91.

[GS99] J P C Greenlees and N P Strickland, *Varieties and local cohomology for chromatic group cohomology rings*, Topology **38** (1999), no. 4, 1093–1139.

[Hai02] R Hain, *The rational cohomology ring of the moduli space of abelian 3-folds*, Math. Res. Lett. **9** (2002), no. 4, 473–491.

[Har71] G Harder, *A Gauss-Bonnet formula for discrete arithmetically defined groups*, Ann. Sci. Éc. Norm. Supér. (4) **4** (1971), 409–455.

[Hen97] H-W Henn, *Centralizers of elementary abelian p-subgroups, the Borel construction of the singular locus and applications to the cohomology of discrete groups*, Topology **36** (1997), no. 1, 271–286.

[Hen99] H-W Henn, *The cohomology of* $\mathrm{SL}(3,\mathbb{Z}[1/2])$, K-Theory **16** (1999), no. 4, 299–359.

[Hir73] F Hirzebruch, *Hilbert modular surfaces*, Série des Conférences de l'Union Mathématique Internationale, No. 4, Université de Genève, Secrétariat de l'Enseignement Mathématique, Geneva, 1973.

[HKR92] M J Hopkins, N J Kuhn, and D C Ravenel, *Morava K-theories of classifying spaces and generalized characters for finite groups*, Algebraic topology (San Feliu de Guíxols, 1990), Lecture Notes in Math., vol. 1509, Springer, Berlin, 1992, pp. 186–209.

[HKR00] ———, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. **13** (2000), no. 3, 553–594.

[Hor05] I Horozov, *Euler characteristics of arithmetic groups*, Math. Res. Lett. **12** (2005), no. 2-3, 275–291.

[HT18] K Hulek and O Tommasi, *The topology of* $\mathcal{A}_g$ *and its compactifications*, Geometry of moduli, Abel Symp., vol. 14, Springer, Cham, 2018, With an appendix by Olivier Taïbi, pp. 135–193.

[HZ86] J Harer and D Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), no. 3, 457–485.

[JM92] S Jackowski and J McClure, *Homotopy decomposition of classifying spaces via elementary abelian subgroups*, Topology **31** (1992), no. 1, 113–132.

[Ji07] L Ji, *Integral Novikov conjectures and arithmetic groups containing torsion elements*, Comm. Anal. Geom. **15** (2007), no. 3, 509–533.

[JW10] L Ji and S A Wolpert, *A cofinite universal space for proper actions for mapping class groups*, In the tradition of Ahlfors-Bers. V, Contemp. Math., vol. 510, Amer. Math. Soc., Providence, RI, 2010, pp. 151–163.

[KLL21] D Kasprowski, K Li and W Lück, *K- and L-theory of graph products of groups*, Groups Geom. Dyn. **15** (2021), no. 1, 269–311.

[KS71] A Karrass and D Solitar, *The free product of two groups with a malnormal amalgamated subgroup*, Canadian J. Math. **23** (1971), 933–959.

[Kaw97] N Kawazumi, *Homology of hyperelliptic mapping class groups for surfaces*, Topology Appl. **76** (1997), no. 3, 203–216.

[LM33] C G Latimer and C C MacDuffee, *A correspondence between classes of ideals and classes of matrices*, Ann. of Math. (2) **34** (1933), no. 2, 313–316.

[Lee94] C-N Lee, *Stable homotopy type of the classifying space of virtually torsion-free groups*, Topology **33** (1994), no. 4, 721–728.

[LS78] R Lee and R H Szczarba, *On the torsion in $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$*, Duke Math. J. **45** (1978), no. 1, 101–129.

[LN98] G Levitt and J-L Nicolas, *On the maximum order of torsion elements in* $\mathrm{GL}(n, \mathbb{Z})$ *and* $\mathrm{Aut}(F_n)$, J. Algebra **208** (1998), no. 2, 630–642.

[LNP22] S Linskens, D Nardin, and L Pol, *Global homotopy theory via partially lax limits*, arXiv:2206.01556, to appear in Geom. Topol.

[Loo93] E Looijenga, *Cohomology of $\mathscr{M}_3$ and $\mathscr{M}_3^1$*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., vol. 150, Amer. Math. Soc., Providence, RI, 1993, pp. 205–228.

[Lur19] J Lurie, *Elliptic Cohomology III: Tempered Cohomology*, 2019, available at author's website `https://www.math.ias.edu/~lurie/papers/Elliptic-III-Tempered.pdf`.

[Lüc02a] W Lück, *Chern characters for proper equivariant homology theories and applications to K- and L-theory*, J. Reine Angew. Math. **543** (2002), 193–234.

[Lüc02b] ———, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002.

[Lüc05a] ———, *Equivariant cohomological Chern characters*, Internat. J. Algebra Comput. **15** (2005), no. 5-6, 1025–1052.

[Lüc05b] ———, *Survey on classifying spaces for families of subgroups*, Infinite groups: geometric, combinatorial and dynamical aspects, Progr. Math., vol. 248, Birkhäuser, Basel, 2005, pp. 269–322.

[Lüc07] ———, *Rational computations of the topological K-theory of classifying spaces of discrete groups*, J. Reine Angew. Math. **611** (2007), 163–187.

[LW12] W Lück and M Weiermann, *On the classifying space of the family of virtually cyclic subgroups*, Pure Appl. Math. Q. **8** (2012), no. 2, 497–555.

[Mis10] G Mislin, *Classifying spaces for proper actions of mapping class groups*, Münster J. Math. **3** (2010), 263–272.

[MKS04] W Magnus, A Karrass, and D Solitar, *Combinatorial group theory. Presentations of groups in terms of generators and relations*, second ed., Dover Publications, Inc., Mineola, NY, 2004,

[Pre73] A Prestel, *Die Fixpunkte der symmetrischen Hilbertschen Modulgruppe zu einem reell-quadratischen Zahlkörper mit Primzahldiskriminante*, Math. Ann. **200** (1973), 123–139.

[Rak15] A Raksit, *Characters in global equivariant homotopy theory*, Undergraduate thesis, Harvard University, 2015.

[Rav82] D C Ravenel, *Morava K-theories and finite groups*, Symposium on Algebraic Topology in honor of José Adem (Oaxtepec, 1981), Contemp. Math., vol. 12, Amer. Math. Soc., Providence, R.I., 1982, pp. 289–292.

[Rez14] C Rezk, *Global homotopy theory and cohesion*, available at author's website `https://faculty.math.illinois.edu/~rezk/global-cohesion.pdf`, 2014.

[Sch06] B Schuster, *Morava k-theory of classifying spaces*, Habilitationsschrift, University of Wuppertal, 2006.

[Ser71] J-P Serre, *Cohomologie des groupes discrets*, Séminaire Bourbaki, 23ème année (1970/1971), Exp. No. 399, 1971, pp. 337–350. Lecture Notes in Math., Vol. 244.

[Ser03] ———, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[SG07] R Sánchez-García, *Equivariant K-homology for some Coxeter groups*, J. Lond. Math. Soc. (2) **75** (2007), no. 3, 773–790.

[SG08] ———, *Bredon homology and equivariant K-homology of* $\mathrm{SL}(3, \mathbb{Z})$, J. Pure Appl. Algebra **212** (2008), no. 5, 1046–1059.

[Sou78] C Soulé, *The cohomology of* $\mathrm{SL}_3(\mathbb{Z})$, Topology **17** (1978), no. 1, 1–22.

[Sta13] N Stapleton, *Transchromatic generalized character maps*, Algebr. Geom. Topol. **13** (2013), no. 1, 171–203.

[Str98] N P Strickland, *Morava E-theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779.

[SY97] D Sjerve and Q Yang, *Conjugacy classes of p-torsion in* $\mathrm{Sp}_{p-1}(\mathbb{Z})$, J. Algebra **195** (1997), no. 2, 580–603.

[Tah71] K-I Tahara, *On the finite subgroups of* $\mathrm{GL}(3, \mathbb{Z})$, Nagoya Math. J. **41** (1971), 169–209.

[TY92] M Tezuka and N Yagita, *Complex K-theory of* $B\mathrm{SL}_3(\mathbb{Z})$, K-Theory **6** (1992), no. 1, 87–95.

[Upa96] S Upadhyay, *Controlled algebraic K-theory of integral group ring of* $\mathrm{SL}(3,\mathbb{Z})$, K-Theory **10** (1996), no. 4, 413–418.

[Wal61] C T C Wall, *Rational Euler characteristics*, Proc. Cambridge Philos. Soc. **57** (1961), 182–184.

[Xia92] Y Xia, *The p-torsion of the Farrell-Tate cohomology of the mapping class group* $\Gamma_{(p-1)/2}$, Topology '90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, de Gruyter, Berlin, 1992, pp. 391–398.

[Yan23] L Yanovski, *Homotopy cardinality via extrapolation of Morava-Euler characteristics*, Selecta Math. (N.S.) **29** (2023), no. 5, Paper No. 81, 36 pp.

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