Graduate Seminar on Global Analysis S4B3
Summer semester 2022

Topic: Chern-Weil theory

The seminar builds on the material covered in the class on *Foundations of analysis and geometry on manifolds* in WS21/22

Summary: Given a smooth manifold M, one can ask for invariants of M and for invariants defined by additional structures for M. Invariants are for example de Rham cohomology classes, and additional structures can be a Riemannian metric or a vector bundle on M.

Chern-Weil theory associates to such a vector bundle of rank k a de Rham cohomology class of degree k, the so-called *Euler class*, and, even more, to a *complex* vector bundle of complex rank k a collection of de Rham cohomology classes of degree 2ℓ for $1 \leq \ell \leq k$, the so-called *Chern classes*. These invariants behave naturally with respect to pull-back and other operations on vector bundles, and they are therefore called characteristic classes. They arise in a large variety of mathematical contexts, like in differential or algebraic geometry, in topology, number theory, theoretical physics and more.

The goal of the seminar is to introduce these and some additional characteristic classes from the viewpoint of principal bundles, study some of their properties and look at examples.

A rough outline of the talks is as follows.

(1) Lie groups and Lie subgroups (Lee p.150-161), examples.
(2) Principal bundles (Dupont Chapter 3)
(3) Extensions and reductions of principal bundles; geometric structures (Dupont Chapter 4, Kobayashi-Nominzu I)
(4) Connections in principal bundles (Dupont Chapter 6)
(5) Lie algebras, connections in a principal bundle (Lee p. 190-199, Dupont p.43-46 of Chapter 6).
(6) The curvature form of a connection (Dupont, part of Chapter 6 and Chapter 7)
(7) Invariant polynomials and the Chern-Weil homomorphism (Dupont, Chapter 9)
(8) Examples of invariant polynomials and characteristic classes (Dupont, Chapter 10, Kobayashi-Nomizu II, p.312-320).
(9) Universal classes: Linear connections, Chern character, Todd genus (Nicolaescu Chapter 8.2)
(10) The Gauss-Bonnet-Chern theorem: Thom class and geometric Euler class (Nicolaescu Chapter 8.3)
(11) Flag manifolds, Grassmannians and their tautological bundles (Bott-Tu p.282-294)

(12) The universal bundle and classification of vector bundles (Bott-Tu p.297-305)

Literature:

(1) R. Bott, L, Tu *Differential forms in algebraic topology*, Springer 1981.
(2) J. Dupont, *Fiber bundles and Chern-Weil theory*, Aarhus 2003.
(3) S. Kobayash, K. Nomizu, *Foundations of differential geometry I,II* Wiley 1969.
(4) J. Lee, *Introduction to smooth manifolds*, Second Edition, Springer 2013,
(5) L. Nicolascu, *Lectures on the geometry of manifolds*, 2018.
(6) L. Tu, Differential geometry: Connections, curvature, and characteristic classes, Springer 2017.

A preliminary meeting will be held via Zoom on Feb. 4 at 12.15h.

Zoom-information: 926 4218 4107
Passcode: chernweil